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Embedding of classical polar unitals in $\mathrm{PG}(2, q^2)$

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A B S T R A C T

A unital, that is, a block-design $2-(q^3+1, q+1, 1)$, is embedded in a projective plane Π of order q^2 if its points and blocks are points and lines of Π. A unital embedded in $\mathrm{PG}(2, q^2)$ is Hermitian if its points and blocks are the absolute points and non-absolute lines of a unitary polarity of $\mathrm{PG}(2, q^2)$. A classical polar unital is a unital isomorphic, as a block-design, to a Hermitian unital. We prove that there exists only one embedding of the classical polar unital in $\mathrm{PG}(2, q^2)$, namely the Hermitian unital.

1. Introduction

In finite geometry, embedding of geometric structures into projective spaces has been a central question for many years which still presents numerous open problems. The most natural one asks about existence and uniqueness, that is, whether a block-design can be embedded in a given projective plane and, if this is the case, in how many different ways such an embedding can be done. In this paper we deal with such a uniqueness problem

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for embedding of the Hermitian unital, as a block design, into a Desarguesian projective plane.

A *unital* is defined to be a set of q^3+1 points equipped with a family of subsets, each of size $q+1$, such that every pair of distinct points is contained in exactly one subset of the family. Such subsets are usually called *blocks*, so unitals are $2-(q^3+1, q+1, 1)$ block-designs. A unital is *embedded* in a projective plane Π of order q^2 if its points are points of Π and its blocks are lines of Π. Sufficient conditions for a unital to be embeddable in a projective plane are given in [7]. Computer aided searches suggest that there should be plenty of unitals, especially for small values of q, but those embeddable in a projective plane are quite rare, see [1,3,9,11,8]. In the Desarguesian projective plane $\mathrm{PG}(2,q^2)$, a unital arises from a unitary polarity in $\mathrm{PG}(2,q^2)$: the points of the unital are the absolute points, and the blocks are the non-absolute lines of the polarity. The name of “Hermitian unital” is commonly used for such a unital since its points are the points of the Hermitian curve defined over $\mathrm{GF}(q^2)$. A *classical polar unital* is a unital isomorphic, as a block-design, to a Hermitian unital. By definition, the classical polar unital can be embedded in $\mathrm{PG}(2,q^2)$ as the Hermitian unital. It has been conjectured for a long time that this is the unique embedding of the classical polar unital in $\mathrm{PG}(2,q^2)$ although no explicit reference seems to be available in the literature. Our goal is to prove this conjecture. Our notation and terminology are standard. The principal references on unitals are [2,5].

2. Projections and Hermitian unital

Let $\mathcal{H}$ be a Hermitian unital in the Desarguesian plane $\mathrm{PG}(2,q^2)$. Any non-absolute line intersects $\mathcal{H}$ in a *Baer subline*, that is a set of $q+1$ points isomorphic to $\mathrm{PG}(1,q)$. Take any two distinct non-absolute lines ℓ and ℓ'. For any point Q outside both ℓ and ℓ', the projection of ℓ to ℓ' from Q takes $\ell\cap\mathcal{H}$ to a Baer subline of ℓ'. We say that Q is a *full point with respect to the line pair* (ℓ,ℓ') if the projection from Q takes $\ell\cap\mathcal{H}$ to $\ell'\cap\mathcal{H}$.

From now on we assume that ℓ and ℓ' meet in a point P of $\mathrm{PG}(2,q^2)$ not lying in $\mathcal{H}$. We denote the polar line of P with respect to the unitary polarity associated to $\mathcal{H}$ by $P^\perp$. Then $P^\perp$ is a non-absolute line. We will prove that if q is even then $P^\perp\cap\mathcal{H}$ contains a unique full point. This does not hold true for odd q. In fact, we will prove that for odd q, $P^\perp\cap\mathcal{H}$ contains zero or two full points depending on the mutual position of ℓ and ℓ'.

To work out our proofs we need some notation and known results regarding $\mathcal{H}$ and the projective unitary group $\mathrm{PGU}(3,q)$ preserving $\mathcal{H}$.

Up to a change of the homogeneous coordinate system (X_1,X_2,X_3) in $\mathrm{PG}(2,q^2)$, the points of $\mathcal{H}$ are those satisfying the equation

$$X_1^{q+1}+X_2^{q+1}+X_3^{q+1}=0. \tag{1}$$

Since the unitary group $\mathrm{PGU}(3,q)$ preserving $\mathcal{H}$ acts transitively on the points of $\mathrm{PG}(2,q^2)$ not lying in $\mathcal{H}$, we may assume $P=(0,1,0)$. Then $P^\perp$ has equation $X_2=0$.