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Finite groups admitting an oriented regular representation

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ABSTRACT

In this paper, we investigate finite groups admitting an oriented regular representation and we give a partial answer to a 1980 question of Lazslo Babai: "Which [finite] groups admit an oriented graph as a DRR?" It is easy to see and well-understood that generalised dihedral groups do not admit ORRs. We prove that, apart from C_3^2 and $C_3 \times C_2^3$, every finite group, which is neither a generalised dihedral group nor a 2-group, has an ORR. In particular, the classification of the finite groups admitting an ORR is reduced to the class of 2-groups.

We also give strong structural conditions on finite 2-groups not admitting an ORR. Finally, based on these results and on some extensive computer computations, we state a conjecture aiming to give a complete classification of the finite groups admitting an ORR.

1. Introduction

All groups and graphs in this paper are finite. Let G be a group and let S be a subset of G. The ***Cayley digraph***, denoted by $\mathrm{Cay}(G, S)$, over G with connection set S is the

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digraph with vertex set G and with (x, y) being an arc if $yx^{-1} \in S$. (An ***arc*** is an ordered pair of adjacent vertices.) Since the group G acts faithfully as a group of automorphisms of $\mathrm{Cay}(G, S)$ via the right regular representation, Cayley digraphs represent groups geometrically and combinatorially as groups of automorphisms of digraphs. Naively, the closer G is to the full automorphism group of $\mathrm{Cay}(G, S)$, the closer this representation is from encoding G graphically.

Following this line of thoughts, it is natural to ask which groups G admit a subset S with G being the automorphism group of $\mathrm{Cay}(G, S)$; that is, $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. We say that G admits a ***digraphical regular representation*** (or DRR for short) if there exists a subset S of G with $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. Babai [1, Theorem 2.1] has given a complete classification of the groups admitting a DRR: except for

$$Q_8,\ C_2^2,\ C_2^3,\ C_2^4 \text{ and } C_3^2, \tag{1}$$

every group admits a DRR.

In light of Babai's result, it is natural to try to combinatorially represent groups as automorphism groups of special classes of Cayley digraphs. Observe that, if S is inverse-closed (that is, $S = \{s^{-1} \mid s \in S\} := S^{-1}$), then $\mathrm{Cay}(G, S)$ is undirected. Now, we say that G admits a ***graphical regular representation*** (or GRR for short) if there exists an inverse-closed subset S of G with $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. With a considerable amount of work culminating in [9,11], the groups admitting a GRR have been completely classified. (The pioneer work of Imrich [12–14] was an important step towards this classification.) It is interesting to observe that, although the classification of the groups admitting a DRR is easier than the classification of the groups admitting a GRR, research and interest first focused on finding GRRs. (In some sense this is natural, occasionally graphs draw more interest than digraphs.) It is also worth noting that various researchers have shown that, for certain families of groups, almost all Cayley graphs are GRRs, or almost all Cayley digraphs are DRRs [3,5,9]. The precise definition of "almost all" is slightly technical and it would take us too far astray to include it in this discussion.

We recall that a ***tournament*** is a digraph $\Gamma = (V, A)$ with vertex set V and arc set A such that, for every two distinct vertices $x, y \in V$, exactly one of (x, y) and (y, x) is in A. After the completion of the classification of DRRs and GRRs, Babai and Imrich [2] proved that every group of odd order except for C_3^2 admits a ***tournament regular representation*** (or TRR for short). That is, each finite odd-order group G different from C_3^2 contains a subset S with $\mathrm{Cay}(G, S)$ being a tournament and with $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. In terms of the connection set S, the Cayley digraph $\mathrm{Cay}(G, S)$ is a tournament if and only if $S \cap S^{-1} = \emptyset$ and $G \setminus \{1\} = S \cup S^{-1}$. This observation makes it clear that a Cayley digraph on G cannot be a tournament if G contains an element of order 2, so only groups of odd order can admit TRRs.

In [1, Problem 2.7], Babai observed that there is one class of Cayley digraphs that is rather interesting and that has not been investigated in the context of regular representations; that is, the class of oriented Cayley digraphs (or as Babai called them, oriented