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Multiplicative structures of the immaculate basis of non-commutative symmetric functions

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ABSTRACT

We continue our development of a new basis for the algebra of non-commutative symmetric functions. This basis is analogous to the basis of Schur functions for the algebra of symmetric functions, and it shares many of its wonderful properties. For instance, in this article we describe non-commutative versions of the Littlewood–Richardson rule and the Murnaghan–Nakayama rule. A surprising relation develops among non-commutative Littlewood–Richardson coefficients, which has implications to the commutative case. Finally, we interpret these new coefficients geometrically as the number of integer points inside a certain polytope.

1. Introduction

Schur functions appear throughout mathematics: as representatives for Schubert classes in the cohomology of the Grassmannian; as characters for the irreducible representations of the symmetric group and of the general linear group; and as an orthonormal basis for the algebra of symmetric functions. The ubiquity of Schur functions makes it an object of central importance in the theory of symmetric functions. Among the char-

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acteristic properties of Schur functions there are combinatorial formulae using tableaux, orthogonality relations, algebraic formulae and each of these may be taken either as the relation defining the basis or a consequence of the definition.

If the algebra of symmetric functions, Sym, is viewed as an algebra with one commutative generator at each degree, then the algebra of non-commutative symmetric functions [9], NSym, is an analogous algebra with one non-commutative generator at each degree. Until relatively recently, the only clear proposal for elements of NSym that are analogous to Schur functions was the ribbon basis.

While this is a natural proposition, researchers have begun to question the assumption that the ribbon basis is the best possible Schur-analogue for NSym and have proposed other bases (each with a different rule for computing the commutative image) [2,7,11]. These are worth exploring as potential tools for resolving some of the positivity, combinatorial and representation theoretical questions in symmetric functions. For instance, Hall–Littlewood symmetric functions seem to have interesting analogues using these bases [2,11] and these are a potential avenue for answering open questions about q, t-Kostka [15] and generalized Kostka coefficients [19].

In [2], the authors proposed a basis, called the immaculate basis, for NSym. This basis is analogous to the basis of Schur functions of Sym through a Jacobi–Trudi-like defining formula as well as a combinatorial formula using composition tableaux, and because immaculate functions indexed by a partition project onto Schur functions indexed by the same partition under the natural map from NSym to Sym. In [3], the authors constructed indecomposable modules for the 0-Hecke algebra whose characters correspond to immaculate functions under the duality between NSym and QSym. This provides further evidence of the virtuousness of the immaculate basis.

The goal of this paper is to further develop the immaculate basis of NSym. The paper is organized as follows. In Section 2 we review the symmetric function theory that we intend to emulate in NSym. In Section 3 we present a brief introduction to NSym and the immaculate basis. In Section 4 we describe an extension of the Pieri rule for immaculate functions (Theorem 4.2) to ribbon shapes (Theorem 4.3). In Section 5 we prove a version of the Murnaghan–Nakayama rule (Theorem 5.1) for immaculate functions. In Section 6 we deduce a dual version of our Murnaghan–Nakayama rule (Corollary 6.2). In Section 7 we formulate and prove an analogue of the Littlewood–Richardson rule for immaculate functions (Theorem 7.3) and discover a relation amongst Littlewood–Richardson coefficients for Schur functions (Corollary 7.6) that is not easily deduced from the theory of symmetric functions. Finally, in Section 8 we provide a geometric interpretation of immaculate Littlewood–Richardson coefficients as the number of integer points inside certain polytopes.

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