Journal of Combinatorial Theory, Series A 152 (2017) 138–158

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Combinatorial and inductive methods for the tropical maximal rank conjecture

David Jensen, Sam Payne

ARTICLE INFO

Article history:
Received 20 September 2016
Available online xxxx

Keywords:
Tropical geometry
Tropical independence
Maximal rank conjecture

ABSTRACT

We produce new combinatorial methods for approaching the tropical maximal rank conjecture, including inductive procedures for deducing new cases of the conjecture on graphs of increasing genus from any given case. Using explicit calculations in a range of base cases, we prove this conjecture for the canonical divisor, and in a wide range of cases for $m = 3$, extending previous results for $m = 2$.

1. Introduction

The classical maximal rank conjecture in algebraic geometry predicts the Hilbert function in each degree m for the general embedding of a general algebraic curve of fixed genus g and degree d in projective space $\mathbb{P}^r$, by specifying that certain linear multiplication maps on global sections should be of maximal rank.

Maximal Rank Conjecture. *Suppose g, r, d, and m are positive integers, with $r \geq 3$, such that $g \geq (r + 1)(g - d + r)$, and let $X \subset \mathbb{P}^r$ be a general curve of genus g and degree d. Then the multiplication map*

E-mail addresses: dave.jensen@uky.edu (D. Jensen), sam.payne@yale.edu (S. Payne).

http://dx.doi.org/10.1016/j.jcta.2017.06.003
0097-3165/

$$\mu_m : \mathrm{Sym}^m H^0(X, \mathcal{O}_X(1)) \to H^0(X, \mathcal{O}_X(m))$$

is either injective or surjective.

In previous work, we introduced the notion of tropical independence to study ranks of such multiplication maps combinatorially, using minima of piecewise-linear functions on graphs arising via tropicalization. As first applications, we gave a new proof of the Gieseker–Petri theorem [10], and formulated a purely combinatorial analogue on tropical curves of the maximal rank conjecture for algebraic curves, which we proved for $m = 2$ [11].

Tropical Maximal Rank Conjecture. *Suppose g, r, d, and m are positive integers, with $r \geq 3$, such that $g \geq (r+1)(g-d+r)$ and $d < g+r$. Then there is a divisor D of rank r and degree d whose class is vertex avoiding on a chain of g loops with generic edge lengths, and a tropically independent subset $\mathcal{A} \subset \{\psi_I \mid |I| = m\}$ of size*

$$|\mathcal{A}| = \min\left\{\binom{r+m}{m},\ md - g + 1\right\}.$$

Note that each case of the tropical maximal rank conjecture implies the classical maximal rank conjecture for the same parameters g, r, d, and m, through well-known tropical lifting and specialization arguments [11, Proposition 4.7].

As the links to algebraic geometry are already established, the main purpose of this paper is to introduce new combinatorial methods for approaching the tropical maximal rank conjecture, and to use these methods to prove the conjecture for canonical divisors (i.e. the case where $r = g-1$ and $d = 2g-2$, for all m), and for a wide range of cases with $m = 3$. Our results include inductive statements through which new cases of the tropical maximal rank conjecture can be deduced from other cases with smaller parameters (Theorems 1.1 and 1.2), along with explicit combinatorial calculations in increasingly intricate examples (see Sections 5–7), providing base cases for applying such inductions.

To state our results as cleanly as possible, we find it helpful to divide the space of parameters (g, r, d, m) into the *injective range*, where $\binom{r+m}{m} \leq md - g + 1$, and the *surjective range*, where $\binom{r+m}{m} \geq md - g + 1$. Under the hypotheses of the classical maximal rank conjecture, when $m > 1$ the vector spaces $\mathrm{Sym}^m H^0(X, \mathcal{O}_X(1))$ and $H^0(X, \mathcal{O}_X(m))$ have dimension $\binom{r+m}{m}$ and $md - g + 1$, respectively, so the injective range (resp. surjective range) is exactly the set of parameters for which the classical maximal rank conjecture predicts μ_m to be injective (resp. surjective). In the setting of the tropical maximal rank conjecture, the set $\{\psi_I \mid \#I = m\}$ has size $\binom{r+m}{m}$, so (g, r, d, m) is in the injective or surjective range, respectively, according to whether the tropically independent set $\mathcal{A}$ is supposed to consist of all possible ψ_I, or a subset of size $md - g + 1$.

We also find it convenient to change coordinates on the space of parameters in the conjectures, setting $s = g - d + r$ and $\rho = g - (r+1)(g - d + r)$. These new parameters