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A divisibility result in combinatorics of generalized braids

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ABSTRACT

For every finite Coxeter group Γ, each positive braid in the corresponding braid group admits a unique decomposition as a finite sequence of elements of Γ, the so-called Garside-normal form. The study of the associated adjacency matrix Adj(Γ) allows to count the number of Garside-normal form of a given length. In this paper we prove that the characteristic polynomial of $\mathrm{Adj}(B_n)$ divides the one of $\mathrm{Adj}(B_{n+1})$. The key point is the use of a Hopf algebra based on signed permutations. A similar result was already known for the type A. We observe that this does not hold for type D. The other Coxeter types (I, E, F and H) are also studied.

0. Introduction

Let S be a set. A *Coxeter matrix* on S is a symmetric matrix $M = (m_{s,t})$ whose entries are in $\mathbb{N} \cup \{+\infty\}$ and such that $m_{s,t} = 1$ if, and only if, $s = t$. A Coxeter matrix is usually represented by a labelled *Coxeter graph* Γ whose vertices are the elements of S; there is an edge between s and t labelled by $m_{s,t}$ if, and only if, $m_{s,t} \geqslant 3$. From such a graph Γ, we define a group W_Γ by the presentation:

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$$W_\Gamma = \left\langle S \;\middle|\; \begin{array}{cc} s^2 = 1 & \text{for } s \in S \\ \mathrm{prod}(s,t;m_{s,t}) = \mathrm{prod}(t,s;m_{t,s}) & \text{for } s,t \in S \text{ and } m_{s,t} \neq +\infty \end{array} \right\rangle,$$

where $\mathrm{prod}(s,t;m_{s,t})$ is the product $s\,t\,s\ldots$ with $m_{s,t}$ terms. The pair (W_Γ, S) is called a *Coxeter system*, and W_Γ is the *Coxeter group* of type Γ. Note that two elements s and t of S commute in W_Γ if, and only if, s and t are not connected in Γ. Denoting by $\Gamma_1, ..., \Gamma_k$ the connected components of Γ, we obtain that W_Γ is the direct product $W_{\Gamma_1} \times ... \times W_{\Gamma_k}$. The Coxeter group W_Γ is said to be *irreducible* if the Coxeter graph Γ is connected. We say that a Coxeter graph is *spherical* if the corresponding group W_Γ is finite. There are four infinite families of connected spherical Coxeter graphs: A_n $(n \geqslant 1)$, B_n $(n \geqslant 2)$, D_n $(n \geqslant 4)$, $I_2(p)$ $(p \geqslant 5)$, and six exceptional graphs E_6, E_7, E_8, F_4, H_3 and H_4. For $\Gamma = A_n$, the group W_Γ is the symmetric group $\mathfrak{S}_{n+1}$.

For a Coxeter graph Γ, we define the *braid group* $B(W_\Gamma)$ by the presentation:

$$B(W_\Gamma) = \left\langle S \;\middle|\; \mathrm{prod}(s,t;m_{s,t}) = \mathrm{prod}(t,s;m_{t,s}) \text{ for } s,t \in S \text{ and } m_{s,t} \neq +\infty \right\rangle,$$

and the positive braid monoid to be the monoid presented by:

$$B^+(W_\Gamma) = \left\langle S \;\middle|\; \mathrm{prod}(s,t;m_{s,t}) = \mathrm{prod}(t,s;m_{t,s}) \text{ for } s,t \in S \text{ and } m_{s,t} \neq +\infty \right\rangle^+.$$

The groups $B(W_\Gamma)$ are known as Artin–Tits groups; they have been introduced in [4,2] and in [10] for spherical type. The embedding of the monoid $B^+(W_\Gamma)$ in the corresponding group $B(W_\Gamma)$ was established by L. Paris in [14]. For $\Gamma = A_n$, the braid group $B(W_{A_n})$ is the Artin braid group B_n and $B^+(W_{A_n})$ is the monoid of positive Artin braids B_n^+.

We now suppose that Γ is a spherical Coxeter graph. The Garside normal form allows us to express each braid β of $B^+(W_\Gamma)$ as a unique finite sequence of elements of W_Γ. This defines an injection Gar form $B^+(W_\Gamma)$ to $W_\Gamma^{(\mathbb{N})}$. The Garside length of a braid $\beta \in B^+(W_\Gamma)$ is the length of the finite sequence $\mathrm{Gar}(\beta)$. If, for all $\ell \in \mathbb{N}$, we denote by $B^\ell(W_\Gamma)$ the set of braids whose Garside length is ℓ, the map Gar defines a bijection between $B^\ell(W_\Gamma)$ and $\mathrm{Gar}(B^+(W_\Gamma)) \cap W_\Gamma^\ell$.

A sequence $(s,t) \in W_\Gamma^2$ is said *normal* if (s,t) belongs to $B^2(W_\Gamma)$. From a local characterization of the Garside normal form, for $\ell \geqslant 2$ the sequence $(w_1, ..., w_\ell)$ of W_Γ^ℓ belongs to $\mathrm{Gar}(B^+(W_\Gamma))$ if, and only if, (w_i, w_{i+1}) is normal for all $i = 1, ..., \ell - 1$. Roughly speaking, in order to recognize the elements of $\mathrm{Gar}(B^+(W_\Gamma))$ among thus of $W_\Gamma^{(\mathbb{N})}$ it is enough to recognize the elements of $B^2(W_\Gamma)$ among thus of W_Γ^2.

We define a square matrix $\mathrm{Adj}_\Gamma = (a_{u,v})$, indexed by the elements of W_Γ, by:

$$a_{u,v} = \begin{cases} 1 & \text{if } (u,v) \text{ is normal,} \\ 0 & \text{otherwise.} \end{cases}$$

For $\ell \geqslant 1$, the number of positive braids whose Garside length is ℓ is then: