Journal of Combinatorial Theory, Series A 152 (2017) 225–254

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Finite flag-transitive affine planes with a solvable automorphism group

Tao Feng

School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, Zhejiang, China

A R T I C L E I N F O

Article history:
Received 25 September 2016
Available online xxxx

Keywords:
Spread
Affine plane
Planar function
Commutative semifield
Quadratic form
Permutation polynomial

A B S T R A C T

In this paper, we consider finite flag-transitive affine planes with a solvable automorphism group. Under a mild number-theoretic condition involving the order and dimension of the plane, the translation complement must contain a linear cyclic subgroup that either is transitive or has two equal-sized orbits on the line at infinity. We develop a new approach to the study of such planes by associating them with planar functions and permutation polynomials in the odd order and even order case respectively. In the odd order case, we characterize the Kantor–Suetake family by using Menichetti's classification of generalized twisted fields and Blokhuis, Lavrauw and Ball's classification of rank two commutative semifields. In the even order case, we develop a technique to study permutation polynomials of DO type by quadratic forms and characterize such planes that have dimensions up to four over their kernels.

1. Introduction

Let V be a $2n$-dimensional vector space over the finite field $\mathbb{F}_q$. A *spread* $\mathcal{S}$ of V is a collection of n-dimensional subspaces that partitions the nonzero vectors in V. The members of $\mathcal{S}$ are the *components*, and V is the *ambient space*. The *kernel* is the subring

E-mail address: tfeng@zju.edu.cn.

http://dx.doi.org/10.1016/j.jcta.2017.06.011

of $\Gamma L(V)$ that fixes each component, and it is a finite field containing $\mathbb{F}_q$. The *dimension* of $\mathcal{S}$ is the common value of the dimensions of its components over the kernel. The *automorphism group* $\mathrm{Aut}(\mathcal{S})$ is the subgroup of $\Gamma L(V)$ that maps components to components. The incidence structure $\Pi(\mathcal{S})$ with point set V and line set $\{W + v : W \in \mathcal{S}, v \in V\}$ and incidence being inclusion is a translation plane. The kernel or dimension of $\Pi(\mathcal{S})$ is that of $\mathcal{S}$ respectively. Andre [4] has shown that $\mathrm{Aut}(\mathcal{S})$ is the translation complement of the plane $\Pi(\mathcal{S})$ and each finite translation plane can be obtained from a spread in this way. Two spreads $\mathcal{S}$ and $\mathcal{S}'$ of V are *isomorphic* if $\mathcal{S}' = \{g(W) : W \in \mathcal{S}\}$ for some $g \in \Gamma L(V)$, and isomorphic spreads correspond to isomorphic planes.

An affine plane is called *flag-transitive* if it admits a collineation group which acts transitively on the flags, namely, the incident point-line pairs. Throughout this paper, we will only consider finite planes. Wagner [46] has shown that finite flag-transitive planes are necessarily translation planes, so the plane must have prime power order and can be constructed from a spread $\mathcal{S}$ with ambient space V of dimension $2n$ over $\mathbb{F}_q$ for some n and q. The affine plane $\Pi(\mathcal{S})$ constructed from a spread $\mathcal{S}$ is flag-transitive if and only if $\mathrm{Aut}(\mathcal{S})$ is transitive on the components. Foulser has determined all flag-transitive groups of finite affine planes in [22,23]. The only non-Desarguesian flag-transitive affine planes with nonsolvable collineation groups are the nearfield planes of order 9, the Hering plane of order 27 [24], and the Lüneburg planes of even order [35], cf. [14,29]. In the solvable case, Foulser has shown that with a finite number of exceptions, which are explicitly described, a solvable flag transitive group of a finite affine plane is a subgroup of a one-dimensional Desarguesian affine plane.

Kantor and Suetake have constructed non-Desarguesian flag-transitive affine planes of odd order in [28,30,42,43], and we will refer to these planes as the Kantor–Suetake family. The dimension two case is also due to Baker and Ebert [8]. Kantor and Williams have constructed large numbers of flag-transitive affine planes of even order arising from symplectic spreads in [26,31]. The dimensions of these planes over their kernels are odd. It remains open whether there is a non-Desarguesian flag-transitive affine plane of even order whose dimension over its kernel is even and greater than 2. Prince has completed the determination of all the flag-transitive affine planes of order at most 125 in [41], and there are only the known ones.

Except for the Lüneburg planes and the Hering plane of order 27, all the known finite non-Desarguesian flag-transitive affine planes have a translation complement which contains a linear cyclic subgroup that either is transitive or has two equal-sized orbits on the line at infinity. Under a mild number-theoretic condition involving the order and dimension of the plane (see Lemma 3.1 below), it can be shown that one of these actions must occur. We call flag-transitive planes of the first kind $\mathcal{C}$*-planes* and those of the second kind $\mathcal{H}$*-planes*, and call the corresponding spreads of *type* $\mathcal{C}$ and *type* $\mathcal{H}$ respectively. There has been extensive study on these two types of planes in the literature. In the case the plane has odd order and dimension two or three over its kernel, it has been shown that the known examples are the only possibilities for either of these two types, see [5–7,9,10,21]. The classification takes a geometric approach by making use of