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Vertex-imprimitive symmetric graphs with exactly one edge between any two distinct blocks

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A graph Γ is called G-symmetric if it admits G as a group of automorphisms acting transitively on the set of ordered pairs of adjacent vertices. We give a classification of G-symmetric graphs Γ with $V(\Gamma)$ admitting a nontrivial G-invariant partition $\mathcal{B}$ such that there is exactly one edge of Γ between any two distinct blocks of $\mathcal{B}$. This is achieved by giving a classification of $(G,2)$-point-transitive and G-block-transitive designs $\mathcal{D}$ together with G-orbits Ω on the flag set of $\mathcal{D}$ such that $G_{\sigma,L}$ is transitive on $L\setminus\{\sigma\}$ and $L\cap N=\{\sigma\}$ for distinct $(\sigma,L),(\sigma,N)\in\Omega$, where $G_{\sigma,L}$ is the setwise stabilizer of L in the stabilizer G_σ of σ in G. Along the way we determine all imprimitive blocks of G_σ on $V\setminus\{\sigma\}$ for every 2-transitive group G on a set V, where $\sigma\in V$.

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1. Introduction

Intuitively, a graph is symmetric if all its arcs have the same status in the graph, where an *arc* is an ordered pair of adjacent vertices. Since Tutte's seminal work [28], symmetric graphs have long been important objects of study in graph theory due to their intrinsic beauty and wide applications (see [26] for an excellent overview of the area). In this paper we give a classification of those symmetric graphs with an automorphism group acting transitively on the arc set and imprimitively on the vertex set such that there is exactly one edge between any two blocks of the underlying invariant partition.

A finite graph Γ with vertex set $V(\Gamma)$ is called *G-symmetric* (or *G-arc-transitive*) if it admits G as a group of automorphisms (that is, G acts on $V(\Gamma)$ and preserves the adjacency relation of Γ) such that G is transitive on $V(\Gamma)$ and transitive on the set of arcs of Γ. (A graph is *symmetric* if it is $\mathrm{Aut}(\Gamma)$-symmetric, where $\mathrm{Aut}(\Gamma)$ is the full automorphism group of Γ.) The group G is said to be imprimitive on $V(\Gamma)$ if $V(\Gamma)$ admits a nontrivial *G-invariant partition* $\mathcal{B} = \{B, C, \ldots\}$, that is, $1 < |B| < |V(\Gamma)|$ and $B^g := \{\alpha^g \mid \alpha \in B\} \in \mathcal{B}$ for any $g \in G$ and $B \in \mathcal{B}$. In this case $(\Gamma, G, \mathcal{B})$ is said to be a *symmetric triple*. The *quotient graph* of Γ relative to $\mathcal{B}$, denoted by $\Gamma_{\mathcal{B}}$, is defined to be the graph with vertex set $\mathcal{B}$ such that $B, C \in \mathcal{B}$ are adjacent if and only if there exists at least one edge of Γ with one end-vertex in B and the other in C. (As usual we assume that $\Gamma_{\mathcal{B}}$ has at least one edge so that each block of $\mathcal{B}$ is an independent set of Γ.) For adjacent $B, C \in \mathcal{B}$, define $\Gamma[B, C]$ to be the bipartite subgraph of Γ with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$, where $\Gamma(B)$ is the set of vertices of Γ with at least one neighbour in B. Since $\Gamma_{\mathcal{B}}$ can be easily seen to be G-symmetric, this bipartite graph is independent of the choice of adjacent B, C up to isomorphism. Denote by $\Gamma_{\mathcal{B}}(B)$ the neighbourhood of B in $\Gamma_{\mathcal{B}}$, and by $\Gamma_{\mathcal{B}}(\alpha)$ the set of blocks of $\mathcal{B}$ containing at least one neighbour of $\alpha \in V(\Gamma)$ in Γ. Denote

$$v := |B|, \quad r := |\Gamma_{\mathcal{B}}(\alpha)|, \quad b := |\Gamma_{\mathcal{B}}(B)|, \quad k := |\Gamma(C) \cap B|. \tag{1}$$

Since Γ is G-symmetric and $\mathcal{B}$ is G-invariant, these parameters are independent of the choice of $\alpha \in V(\Gamma)$ and adjacent $B, C \in \mathcal{B}$.

Various possibilities for $\Gamma[B, C]$ can happen. In the "densest" case where $\Gamma[B, C] \cong K_{v,v}$ is a complete bipartite graph, Γ is uniquely determined by $\Gamma_{\mathcal{B}}$, namely, $\Gamma \cong \Gamma_{\mathcal{B}}[K_v]$ is the lexicographic product of $\Gamma_{\mathcal{B}}$ by the complete graph K_v. The "sparsest" case where $\Gamma[B, C] \cong K_2$ (that is, $k = 1$) can also happen; in this case Γ is called a *spread* of $\Gamma_{\mathcal{B}}$ in [16], where it was shown that spreads play a significant role in the study of edge-primitive graphs. Spreads have also arisen from some other classes of symmetric graphs (see [24, 32,30]), and a study of them was undertaken in [31, Section 4]. Spreads of cycles and complete graphs with $r = 1$ were briefly discussed in [15, Section 4], where Gardiner and Praeger remarked that when $k = 1$ and $\Gamma_{\mathcal{B}}$ is a complete graph "it is not at all clear what one can say about Γ in general".