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Catching a fast robber on the grid

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A B S T R A C T

We study the problem of cops and robbers on the grid where the robber is allowed to move faster than the cops. It is well known that two cops are necessary and sufficient to catch the robber on any finite grid when the robber has unit speed. Here, we prove that if the speed of the robber exceeds a sufficiently large absolute constant, then the number of cops needed to catch the robber on an $n \times n$ grid is $\exp(\Omega(\log n / \log\log n))$.

1. Introduction

The game of *Cops and Robbers*, introduced almost thirty years ago independently by Nowakowski and Winkler [14] and Quilliot [15], is a perfect information pursuit-evasion game played on an undirected graph G as follows. There are two players, a set of cops and one robber. The game begins with the cops being placed onto vertices of their choice in G and then the robber, being fully aware of the placement of the cops, positions himself

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at a vertex of his choosing. Afterwards, they move alternately, first the cops and then the robber along the edges of the graph G. In the cops' turn, each cop may move to an adjacent vertex, or remain where he is, and similarly for the robber; also, multiple cops are allowed to occupy the same vertex. The cops win if at some time there is a cop at the same vertex as the robber; otherwise, the robber wins. The minimum number of cops for which the cops have a winning strategy, no matter how the robber plays, is called the *cop number* of G.

Perhaps the most well known problem concerning the game of cops and robbers is Meyniel's conjecture which asserts that $O(\sqrt{n})$ cops are sufficient to catch the robber on any n-vertex graph. While Meyniel's conjecture has attracted a great deal of attention, progress towards the conjecture in its full generality has been rather slow; see [2] for a broad overview, and [9,16] for the state of the art.

In this note, we shall be concerned with a variant of the question where the robber is allowed to move faster than the cops. Let us suppose that the cops move normally as before while the robber is allowed to move at speed $R \in \mathbb{N}$; in other words, the robber may, on his turn, take any walk of length at most R from his current position that does not pass through a vertex occupied by a cop. The definition of the cop number in this setting is analogous. This variant was originally considered by Fomin, Golovach, Kratochvíl, Nisse and Suchan [5] and following them, Frieze, Krivelevich and Loh [6], Mehrabian [12], and Alon and Mehrabian [1] have obtained results about how large the cop number of an n-vertex graph can be when the robber has a fixed speed $R > 1$.

It is natural to ask how the cop number of a given graph changes, if at all, when the speed of the robber increases from 1 to some $R > 1$. The most natural example of a graph where this question is interesting is the $n \times n$ grid of squares where two squares of the grid are adjacent if and only if they share an edge. Let us write $f_R(n)$ for the minimum number of cops needed to catch a robber of speed R on an $n \times n$ grid. Maamoun and Meyniel [10] showed, amongst other things, that $f_1(n) = 2$ for all $n \geq 2$. However, the flavour of the problem changes completely as soon as the robber is allowed to move faster than the cops. Nisse and Suchan [13] showed that $f_2(n) = \Omega(\sqrt{\log n})$. Our aim in this note is to prove the following extension.

Theorem 1.1. *There exists an $R \in \mathbb{N}$ and a $c_R > 0$ such that for all sufficiently large $n \in \mathbb{N}$, we have*

$$f_R(n) \geq \exp\left(\frac{c_R \log n}{\log \log n}\right).$$

To keep the presentation simple, we shall make no attempt to optimise the speed of the robber; we prove Theorem 1.1 with $R = 10^{25}$.

Note that $f_R(n) \leq n$ for every $R \in \mathbb{N}$ since n cops can catch a robber of any speed on the $n \times n$ grid by lining up on the bottom edge of the grid and then marching upwards together. We suspect that this trivial upper bound is closer to the truth than Theorem 1.1; we conjecture the following.