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The maximum size of a partial spread in a finite projective space

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ABSTRACT

Let n and t be positive integers with $t < n$, and let q be a prime power. A *partial* $(t-1)$*-spread* of $\mathrm{PG}(n-1,q)$ is a set of $(t-1)$-dimensional subspaces of $\mathrm{PG}(n-1,q)$ that are pairwise disjoint. Let $r = n \bmod t$ and $0 \le r < t$. We prove that if $t > (q^r-1)/(q-1)$, then the maximum size, i.e., cardinality, of a partial $(t-1)$-spread of $\mathrm{PG}(n-1,q)$ is $(q^n - q^{t+r})/(q^t-1)+1$. This essentially settles a main open problem in this area. Prior to this result, this maximum size was only known for $r \in \{0,1\}$ and for $r = q = 2$.

1. Introduction

Let n and t be positive integers with $t < n$, and let q be a prime power. Let $\mathrm{PG}(n-1,q)$ denote the $(n-1)$-dimensional projective space over the finite field $\mathbb{F}_q$. A *partial* $(t-1)$*-spread* S of $\mathrm{PG}(n-1,q)$ is a collection of $(t-1)$-dimensional subspaces of $\mathrm{PG}(n-1,q)$ that are pairwise disjoint. If S contains all the points of $\mathrm{PG}(n-1,q)$, then it is called a $(t-1)$*-spread*. It follows from the work of André [2] (also see [7, p. 29]) that a $(t-1)$-spread of $\mathrm{PG}(n-1,q)$ exists if and only if t divides n.

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Given positive integers n and t with $t < n$, the problem of finding the maximum size, i.e., cardinality, of a partial $(t-1)$-spread of $\mathrm{PG}(n-1, q)$ is rather a natural one. It is directly related to the general problem of classifying the *maximal* partial $(t-1)$-spread. A maximal partial $(t-1)$-spread is a set of pairwise disjoint $(t-1)$-dimensional subspaces which cannot be extended to a larger set. This problem has been extensively studied [10, 18,21,27]. Besides their traditional relevance to Galois geometry, partial $(t-1)$-spreads are used to build byte-correcting codes (e.g., see [13,25]), 1-perfect mixed error-correcting codes (e.g., see [20,24]), orthogonal arrays and (s, k, λ)-nets (e.g., see [8]). More recently, partial $(t-1)$-spreads have also attracted renewed attention since they can be viewed as *subspace codes*. In Section 4, we shall say more about the connection between our results and subspace codes.

Let $\mu_q(n, t)$ denote the maximum size of any partial $(t-1)$-spread of $\mathrm{PG}(n-1, q)$. The problem of determining $\mu_q(n, t)$ is a long standing open problem. A general upper bound for $\mu_q(n, t)$ is given by the following theorem of Drake and Freeman [8].

Theorem 1. *Let $r = n \bmod t$ and $0 \le r < t$. Then*

$$\mu_q(n, t) \le \frac{q^n - q^r}{q^t - 1} - \lfloor \omega \rfloor - 1,$$

where $2\omega = \sqrt{4q^t(q^t - q^r) + 1} - (2q^t - 2q^r + 1)$.

The following result is due to André [2] for $r = 0$. For $r = 1$, it is due to Hong and Patel [25] when $q = 2$, and Beutelspacher [3] when $q > 2$.

Theorem 2. *Let $r = n \bmod t$ and $0 \le r < t$. Then*

$$\mu_q(n, t) \ge \frac{q^n - q^{t+r}}{q^t - 1} + 1,$$

where equality holds if $r \in \{0, 1\}$.

In light of Theorem 2, it was conjectured (e.g., see [9,25]) that the value of $\mu_q(n, t)$ is given by the lower bound in Theorem 2. However, this conjecture was disproved by El-Zanati et al. [11] who proved the following result.

Theorem 3. *If $n \ge 8$ and $n \bmod 3 = 2$, then* $\mu_2(n, 3) = \dfrac{2^n - 2^5}{7} + 2$.

Very recently, Kurz [29] proved the following theorem which upholds the lower bound for $\mu_q(n, t)$ when $q = 2$, $r = 2$, and $t > 3$.

Theorem 4. *If $n > t > 3$ and $n \bmod t = 2$, then*

$$\mu_2(n, t) = \frac{2^n - 2^{t+2}}{2^t - 1} + 1.$$