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A proof of the Square Paths Conjecture

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A B S T R A C T

The modified Macdonald polynomials, introduced by Garsia and Haiman (1996), have many astounding combinatorial properties. One such class of properties involves applying the related ∇ operator of Bergeron and Garsia (1999) to basic symmetric functions. The first discovery of this type was the (recently proven) Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov (2005), which relates the expression ∇e_n to parking functions. In (2007), Loehr and Warrington conjectured a similar expression for ∇p_n which is known as the Square Paths Conjecture.

Haglund and Loehr (2005) introduced the notion of schedules to enumerate parking functions with a fixed set of cars in each diagonal. In this paper, we extend the notion of schedules and some related results of Hicks (2013) to labeled square paths. We then apply our new results to prove the Square Paths Conjecture.

1. Introduction

This paper addresses the interplay between symmetric function theory and combinatorics. In particular, we prove that ∇p_n can be expressed as a weighted sum of certain

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labeled lattice paths (called preference functions or labeled square paths). This formula for ∇p_n was originally conjectured by Loehr and Warrington [13]. Here p_n is the nth power symmetric function and ∇ is the symmetric function operator introduced by Bergeron and Garsia [1]. This linear operator is defined by its action on the modified Macdonald polynomials (∇'s eigenfunctions). The Macdonald polynomials are a basis for the ring of symmetric functions first introduced by Macdonald [14] and later modified by Garsia and Haiman [6].

The ∇ operator is also a component of the Shuffle Conjecture. The symmetric function side of the Shuffle Conjecture – ∇ applied to the elementary symmetric functions e_n – was first studied because of its relation to the module of Diagonal Harmonics. In [8], Haglund, Haiman, Loehr, Remmel, and Ulyanov conjectured a combinatorial formula for ∇e_n as an enumeration of certain labeled Dyck paths, called parking functions. This conjecture was refined by Haglund, Morse, and Zabrocki [10] and their refinement was recently proved by Carlsson and Mellit [3].

Definition 1.1. Let $n \geq 1$. A *square path* is a North–East path from $(0, 0)$ to (n, n) which ends with an East step. A *preference function* is a square path endowed with column-increasing labels adjacent to North steps. The labels are known as *cars*. A *Dyck path* is a square path which stays weakly above the line $y = x$, and a preference function whose underlying path is a Dyck path is called a *parking function*.

These two classes of labeled lattice paths – parking functions and preference functions – were introduced by Konheim and Weiss [12] in 1966 in another form. They define a preference function to be a map $f : [n] \to [n]$. For convenience, we will write it as the vector $(f(1), f(2), \ldots, f(n))$. Furthermore, they define a parking function to be a preference function such that $|f^{-1}([k])| \geq k$ for all $1 \leq k \leq n$. Konheim and Weiss motivated this definition by describing a parking procedure in which n cars try to park in n spaces on a one-way street according to a preference function f. The cars will all succeed in parking if and only if the preference function is a parking function.

For our purposes, it is more helpful to think of the lattice-path interpretations which were first introduced by Garsia and Haiman [5]. They give the following bijective correspondences between these and the maps of Konheim and Weiss. Start with an empty $n \times n$ lattice. Write each car which prefers spot 1 (each $i \in f^{-1}(1)$) in column 1, starting at the bottom, from smallest to largest. Then move to the lowest empty row and write all the cars which prefer spot 2 ($f^{-1}(2)$) in column 2 from smallest to largest and bottom to top. Continue this procedure until all the cars have been recorded. Then draw in the unique smallest lattice path which consists of North and East steps and stays above each car. See Fig. 1.

We can also represent preference functions by pairs of sequences as follows.