Journal of Combinatorial Theory, Series A 152 (2017) 380–397

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Additively irreducible sequences in commutative semigroups

Guoqing Wang

Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, PR China

ARTICLE INFO

Article history:
Received 26 September 2015
Available online xxxx

Keywords:
Irreducible sequences
The Davenport constant
Noetherian semigroups
Finite principal ideal rings
Schützenberger groups

ABSTRACT

Let $\mathcal{S}$ be a commutative semigroup, and let T be a sequence of terms from the semigroup $\mathcal{S}$. We call T an (additively) *irreducible* sequence provided that no sum of some of its terms vanishes. Given any element a of $\mathcal{S}$, let $\mathrm{D}_a(\mathcal{S})$ be the largest length of an irreducible sequence such that the sum of all terms from the sequence is equal to a. In the case that any ascending chain of principal ideals starting from the ideal (a) terminates in $\mathcal{S}$, we find necessary and sufficient conditions for $\mathrm{D}_a(\mathcal{S})$ to be finite, and in particular, we give sharp lower and upper bounds for $\mathrm{D}_a(\mathcal{S})$ in case $\mathrm{D}_a(\mathcal{S})$ is finite. We also apply the result to commutative unitary rings. As a special case, the value of $\mathrm{D}_a(\mathcal{S})$ is determined when $\mathcal{S}$ is the multiplicative semigroup of any finite commutative principal ideal unitary ring.

1. Notation and terminology

Let $\mathcal{S}$ be a commutative semigroup. The operation of the semigroup $\mathcal{S}$ is denoted by "+". We say $\mathcal{S}$ is commutative to mean that $a + b = b + a$ holds for any elements $a, b \in \mathcal{S}$. The identity element of $\mathcal{S}$, denoted $0_{\mathcal{S}}$ (if exists), is the unique element e of

E-mail address: gqwang1979@aliyun.com.

http://dx.doi.org/10.1016/j.jcta.2017.07.001
0097-3165/

$\mathcal{S}$ such that $e + a = a$ for every $a \in \mathcal{S}$. The zero element of $\mathcal{S}$, denoted $\infty_{\mathcal{S}}$ (if exists), is the unique element z of $\mathcal{S}$ such that $z + a = z$ for every $a \in \mathcal{S}$. If $\mathcal{S}$ has an identity element $0_{\mathcal{S}}$, we call $\mathcal{S}$ a monoid and let $\mathrm{U}(\mathcal{S}) = \{a \in \mathcal{S} : a + a' = 0_{\mathcal{S}} \text{ for some } a' \in \mathcal{S}\}$ be the group of units of $\mathcal{S}$. Let

$$\mathcal{S}^0 = \begin{cases} \mathcal{S}, & \text{if } \ \mathcal{S} \text{ has an identity element;} \\ \mathcal{S} \cup \{0\}, & \text{if } \ \mathcal{S} \text{ does not have an identity element,} \end{cases}$$

be the monoid obtained by adjoining an identity element to $\mathcal{S}$ only when necessary. Let $\mathcal{S}^\bullet = \mathcal{S}^0 \setminus \{0_{\mathcal{S}}\}$. For any subset $A \subseteq \mathcal{S}$, let $\mathrm{St}(A) = \{c \in \mathcal{S} : c + a \in A \text{ for every } a \in A\}$ be the stabilizer of the set A in the semigroup $\mathcal{S}$, which is a subsemigroup of $\mathcal{S}$. For any element $a \in \mathcal{S}$, let $(a) = \{a + c : c \in \mathcal{S}^0\}$ denote the *principal ideal* generated by the element $a \in \mathcal{S}$. Then **Green's preorder** on the semigroup $\mathcal{S}$, denoted $\leqq_{\mathcal{H}}$, is defined by $a \leqq_{\mathcal{H}} b \Leftrightarrow a = b$ or $a = b + c$ for some $c \in \mathcal{S}$, equivalently, $(a) \subseteq (b)$. Green's congruence, denoted $\mathcal{H}$, is a basic relation introduced by Green for semigroups which is defined by: $a \ \mathcal{H} \ b \Leftrightarrow a \leqq_{\mathcal{H}} b$ and $b \leqq_{\mathcal{H}} a \Leftrightarrow (a) = (b)$. For any element a of $\mathcal{S}$, let H_a be the congruence class by $\mathcal{H}$ containing a. We write $a <_{\mathcal{H}} b$ to mean that $a \leqq_{\mathcal{H}} b$ but $H_a \neq H_b$. In what follows, R is always a commutative unitary ring, and $\mathcal{S}_R$ the multiplicative semigroup of the ring R.

Let Λ be any partially ordered set. We say Λ has the ascending chain condition (a.c.c.), provided that any ascending chain terminates. We say a commutative semigroup $\mathcal{S}$ (a commutative ring R) satisfies a.c.c. for principal ideals, or for ideals, or for congruences provided that the above corresponding partially ordered set Λ has the a.c.c., where Λ denotes the partially ordered set consisting of principal ideals, or of ideals, or of congruences, in $\mathcal{S}$ (in R) formed by inclusions, respectively. A commutative semigroup $\mathcal{S}$ is said to be **Noetherian** provided that the semigroup $\mathcal{S}$ satisfies the a.c.c. for congruences.

Let a be an element of any commutative semigroup $\mathcal{S}$, or an element of any commutative unitary ring R. We define $\Psi(a)$ to be the largest length $\ell \in \mathbb{N}_0 \cup \{\infty\}$ of strictly ascending principal ideals chain of $\mathcal{S}^0$ (of R accordingly) starting from (a), i.e., the largest $\ell \in \mathbb{N}_0 \cup \{\infty\}$ such that there exist ℓ elements $a_1, a_2, \ldots, a_\ell \in \mathcal{S}^0$ ($a_1, a_2, \ldots, a_\ell \in R$ respectively) with $(a) \subsetneq (a_1) \subsetneq \cdots \subsetneq (a_\ell)$. When $a \in \mathcal{S}$, $\Psi(a)$ can be equivalently defined as the largest length of a strictly ascending Green's preorder chain starting from a:

$$a <_{\mathcal{H}} a_1 <_{\mathcal{H}} \cdots <_{\mathcal{H}} a_\ell.$$

Since the commutative ring R is unitary, we see that the principal ideals of the ring R coincide with the principal ideals of the semigroup $\mathcal{S}_R$, and therefore, the definition $\Psi(a)$ is consistent no matter whether we regard a as the element of the ring R or as the element of the multiplicative semigroup $\mathcal{S}_R$.

We introduce the definition of **Schützenberger group** which plays a key role in giving the main results of this manuscript. Each $c \in \mathrm{St}(H_a)$ induces a mapping $\gamma_c : H_a \to H_a$