Journal of Combinatorial Theory, Series A 150 (2017) 1–35

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Cutting algebraic curves into pseudo-segments and applications

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ARTICLE INFO

Article history:
Received 16 May 2016
Available online xxxx

Keywords:
Depth cycles
Polynomial method
Polynomial partitioning
Pseudo-segments
Incidence geometry
Lenses
Levels in arrangements
Marked faces in arrangements

ABSTRACT

We show that a set of n algebraic plane curves of constant maximum degree can be cut into $O(n^{3/2} \operatorname{polylog} n)$ Jordan arcs, so that each pair of arcs intersect at most once, i.e., they form a collection of *pseudo-segments*. This extends a similar (and slightly better) bound for pseudo-circles due to Marcus and Tardos. Our result is based on a technique of Ellenberg, Solymosi and Zahl that transforms arrangements of plane curves into arrangements of space curves, so that lenses (pairs of subarcs of the curves that intersect at least twice) become vertical depth cycles. We then apply a variant of a technique of Aronov and Sharir to eliminate these depth cycles by making a small number of cuts, which corresponds to a small number of cuts to the original planar arrangement of curves. After these cuts have been performed, the resulting curves form a collection of pseudo-segments.

Our cutting bound leads to new incidence bounds between points and constant-degree algebraic curves. The conditions for these incidence bounds are slightly stricter than those for the current best-known bound of Pach and Sharir; for our result to hold, the curves must be algebraic and of bounded maximum degree, while Pach and Sharir's bound only imposes weaker, purely topological constraints on the curves. However,

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[1] Supported by Grant 2012/229 from the U.S.–Israel Binational Science Foundation, by Grant 892/13 from the Israel Science Foundation, by the Israeli Centers for Research Excellence (I-CORE) program (center no. 4/11), and by the Hermann Minkowski–MINERVA Center for Geometry at Tel Aviv University.

[2] Supported by an NSF postdoctoral fellowship.

http://dx.doi.org/10.1016/j.jcta.2017.02.006

when our conditions hold, the new bounds are superior for almost all ranges of parameters. We also obtain new bounds on the complexity of a single level in an arrangement of constant-degree algebraic curves, and a new bound on the complexity of many marked faces in an arrangement of such curves.

1. Introduction

Let Γ be a finite set of curves in $\mathbb{R}^2$. The *arrangement* $\mathcal{A}(\Gamma)$ of Γ is the planar subdivision induced by Γ. Its vertices are the intersection points and the endpoints of the curves of Γ, its edges are the maximal (relatively open) connected subsets of curves in Γ not containing a vertex, and its faces are maximal (open) connected subsets of $\mathbb{R}^2 \setminus \bigcup_{\gamma\in\Gamma}\gamma$. Because of their rich geometric structure and numerous applications, arrangements of curves, and especially of lines and segments, have been widely studied. See [27] for a comprehensive survey.

A *Jordan arc* is the homeomorphic image of the open interval[3] $(0,1)$; unless otherwise specified, all such arcs will be in $\mathbb{R}^2$. We say that a set Γ of Jordan arcs is a set of *pseudo-segments* if every pair of arcs in Γ intersect at most once, and the arcs cross properly at the point of intersection. Note that in this paper, pseudo-segments may be unbounded. Many combinatorial results on arrangements of lines or segments extend to arrangements of pseudo-segments. Three notable examples are (i) the complexity of a single level in an arrangement, (ii) the number of incidences between points and curves in the arrangement, and (iii) the complexity of many (marked) faces in an arrangement; see, e.g., [2,11,29].

However, when two curves are allowed to intersect more than once, the resulting complexity bounds become weaker. One strategy to address this issue is to cut each curve into several pieces so that the resulting pieces form a collection of pseudo-segments, and then apply the existing bounds for pseudo-segments to the resulting collection. If each pair of curves intersect at most E times, then it is always possible to cut n such curves into at most En^2 pieces, so that each pair of pieces intersect at most once. When one does this, however, the resulting complexity bounds for problems (i)–(iii) are generally poor. In order to obtain better bounds, one must cut the curves into fewer pieces.

This strategy has been pursued successfully for the past 20 years; see [2–5,9,11,12, 26,30]. However, previous work has almost exclusively focused on arrangements where each pair of curves can intersect at most twice (sets of curves of this type are called *pseudo-circles*, or, if unbounded, *pseudo-parabolas*). The current best result in this direc-

[3] Sometimes in the literature a Jordan arc is defined to be the homeomorphic image of the closed interval $[0,1]$. In this paper, however, we will always use open intervals.