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Compatibility fans for graphical nested complexes ☆

Thibault Manneville [a], Vincent Pilaud [b]

[a] *LIX, École Polytechnique, Palaiseau, France*
[b] *CNRS & LIX, École Polytechnique, Palaiseau, France*

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A B S T R A C T

Graph associahedra generalize classical associahedra. They realize the nested complex of a graph G, *i.e.* the simplicial complex whose vertices are the tubes (connected induced subgraphs) of G and whose faces are the tubings (collections of pairwise nested or non-adjacent tubes) of G. The constructions of M. Carr and S. Devadoss, of A. Postnikov, and of A. Zelevinsky for graph associahedra are all based on the nested fan, which coarsens the normal fan of the permutahedron. In view of the variety of fan realizations of associahedra, it is tempting to look for alternative fans realizing graphical nested complexes. Motivated by the analogy between finite type cluster complexes and graphical nested complexes, we transpose S. Fomin and A. Zelevinsky's compatibility fans from the former to the latter setting. We define a compatibility degree between two tubes of a graph G and show that the compatibility vectors of all tubes of G with respect to an arbitrary maximal tubing on G support a fan realizing the nested complex of G. When G is a path, we recover F. Santos' Catalan many realizations of the associahedron.

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E-mail addresses: thibault.manneville@lix.polytechnique.fr (T. Manneville), vincent.pilaud@lix.polytechnique.fr (V. Pilaud).

URLs: http://www.lix.polytechnique.fr/~manneville/ (T. Manneville), http://www.lix.polytechnique.fr/~pilaud/ (V. Pilaud).

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1. Introduction

Associahedra. The n-dimensional *associahedron* is a simple polytope whose $\frac{1}{n+2}\binom{2n+2}{n+1}$ vertices correspond to Catalan objects (triangulations of an $(n+3)$-gon, binary trees on $n+1$ nodes, ...) and whose edges correspond to mutations between them (diagonal flips, edge rotations, ...). Its combinatorial structure appeared in early works of D. Tamari [50] and J. Stasheff [48], and was first realized as a convex polytope by M. Haiman [33] and C. Lee [30]. Since then, the associahedron has motivated a flourishing research trend with rich connections to combinatorics, geometry and algebra: polytopal constructions [31,23,5,29], Tamari and Cambrian lattices [34,42,43], diameter and Hamiltonicity [47,11,41,26], geometric properties [1,25,39], combinatorial Hopf algebras [32,21,6,8], to cite a few. The associahedron was also generalized in several directions, in particular to secondary and fiber polytopes [20,2], graph associahedra and nestohedra [3,12,40,15,52, 36], pseudotriangulation polytopes [46], cluster complexes and generalized associahedra [18,7,24,49,22], and brick polytopes [37,38].

Graph associahedra. This paper deals with graph associahedra, which were defined by M. Carr and S. Devadoss [3] in connection to C. De Concini and C. Procesi's wonderful arrangements [9]. Given a simple graph G with κ connected components and $n+\kappa$ vertices, the G-*associahedron* $\mathsf{Asso}(\mathrm{G})$ is an n-dimensional simple polytope whose combinatorial structure encodes the connected subgraphs of G and their nested structure. More precisely, the face lattice of the polar of the G-associahedron is isomorphic to the *nested complex* $\mathsf{Nest}(\mathrm{G})$ on G, defined as the simplicial complex of all collections of tubes (connected induced subgraphs) of G which are pairwise compatible (either nested, or disjoint and non-adjacent). As illustrated in Figs. 1, 2 and 3, the graph associahedra of certain special families of graphs happen to coincide with well-known families of polytopes: classical associahedra are path associahedra, cyclohedra are cycle associahedra, and permutahedra are complete graph associahedra. The graph associahedra were extended to the *nestohedra*, which are simple polytopes realizing the nested complex of arbitrary building sets [40,15]. Graph associahedra and nestohedra have been geometrically realized in different ways: by successive truncations of faces of the standard simplex [3], as Minkowski sums of faces of the standard simplex [40,15], or from their normal fans by exhibiting explicit inequality descriptions [12,52]. For a given graph G, the resulting polytopes all have the same normal fan which coarsens the type A Coxeter arrangement: its rays are the characteristic vectors of the tubes, and its cones are generated by characteristic vectors of compatible tubes. Alternative realizations of graph associahedra with different normal fans are obtained by successive truncations of faces of the cube in [51,13]. The objective of this paper is to provide a new unrelated family of complete simplicial fans realizing the graphical nested complex $\mathsf{Nest}(\mathrm{G})$ for any graph G.

Cluster algebras and cluster fans. Our construction is directly inspired from combinatorial and geometric properties of finite type cluster algebras and generalized associahedra introduced by S. Fomin and A. Zelevinsky in [16–18]. Note that A. Zelevinsky [52] already

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