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The equivariant Kazhdan–Lusztig polynomial of a matroid

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A B S T R A C T

We define the equivariant Kazhdan–Lusztig polynomial of a matroid equipped with a group of symmetries, generalizing the nonequivariant case. We compute this invariant for arbitrary uniform matroids and for braid matroids of small rank.

1. Introduction

The Kazhdan–Lusztig polynomial $P_M(t) \in \mathbb{Z}[t]$ of a matroid M was introduced in [3]. In the case where M is realizable by a linear space $V \subset \mathbb{C}^n$, the coefficient of t^i in $P_M(t)$ is equal to the dimension of the intersection cohomology group $IH^{2i}(X_V; \mathbb{C})$, where X_V is the "reciprocal plane" of V [3, Proposition 3.12]. In particular, this implies that $P_M(t) \in \mathbb{N}[t]$ whenever M is realizable. We conjectured [3, Conjecture 2.3] that $P_M(t) \in \mathbb{N}[t]$ for every matroid M. We also gave some computations of $P_M(t)$ for uniform matroids and braid matroids of small rank.

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The purpose of this paper is to define a more refined invariant. Given a matroid M equipped with an action of a finite group W, we define the equivariant Kazhdan–Lusztig polynomial $P_M^W(t)$. The coefficients of this polynomial are not integers, but rather virtual representations of the group W. If W is the trivial group, the ring of virtual representations of W is $\mathbb{Z}$, and $P_M^W(t)$ is equal to the ordinary polynomial $P_M(t)$. More generally, the polynomial $P_M(t)$ may be obtained from $P_M^W(t)$ by sending a virtual representation to its dimension. If M is equivariantly realizable by a linear space $V \subset \mathbb{C}^n$, the coefficient of t^i in $P_M^W(t)$ is equal to the intersection cohomology group $IH^{2i}(X_V;\mathbb{C})$, regarded as a representation of W (Corollary 2.12). In particular, this implies that the coefficients of $P_M^W(t)$ are honest (rather than virtual) representations of W whenever M is equivariantly realizable. We conjecture that this is the case even in the non-realizable case (Conjecture 2.13). We compute the coefficients of $P_M^W(t)$ for arbitrary uniform matroids (Theorem 3.1) and for braid matroids of small rank (Section 4.3).

It is reasonable to ask why bother with an equivariant version of this invariant, especially since there are still many things that we do not understand about the nonequivariant version. We have four answers to this question, all of which are illustrated by the case of uniform matroids. To set notation, let $U_{m,d}$ be the uniform matroid of rank d on a set of $m+d$ elements, which is equipped with a natural action of the symmetric group S_{m+d}. Let $C_{i,m,d}$ be the coefficient of t^i in the equivariant Kazhdan–Lusztig polynomial of $U_{m,d}$, and let $c_{i,m,d} = \dim C_{i,m,d}$ be the coefficient of t^i in the nonequivariant Kazhdan–Lusztig polynomial.

- **Nicer formulas:** Our formula for $C_{i,m,d}$ (Theorem 3.1) is very simple; it is a multiplicity-free sum of irreducible representations that are easy to describe. We could of course use the hook-length formula for the dimension of an irreducible representation of S_{m+d} to derive a formula for $c_{i,m,d}$, but the resulting formula is messy and unenlightening. Indeed, we computed a table in the appendix of [3] consisting of the numbers $c_{i,m,d}$ for small values of i, m, and d, and at that time we were unable even to guess the general formula. It was only by keeping track of the extra structure that we were able to see the essential pattern.
- **More powerful tools:** After we figured out the correct statement of Theorem 3.1, we attempted to prove the formula for $c_{i,m,d}$ directly (without going through Theorem 3.1), and we failed. The Schubert calculus techniques that we employ in the proof of Theorem 3.1 are considerably more powerful than the tools to which we have access in the nonequivariant setting.
- **Representation stability:** The sequence of representations $C_{i,m,d}$ is uniformly representation stable in the sense of Church and Farb [2], which essentially means that it admits a description that is independent of d, provided that $d \geq m+2i$ (Remark 3.6). This phenomenon cannot be seen by looking at the numbers $c_{i,m,d}$.
- **Non-realizable examples:** It is difficult to write down examples of non-realizable irreducible matroids for which we can compute the Kazhdan–Lusztig polynomial, and therefore we had no nontrivial checks of our non-negativity conjecture in the non-