Journal of Combinatorial Theory, Series A 150 (2017) 377–385

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Ehrhart quasi-period collapse in rational polygons ☆

Tyrrell B. McAllister, Matthew Moriarity

Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA

ARTICLE INFO

Article history:
Received 4 March 2015
Available online 6 April 2017

Keywords:
Ehrhart polynomials
Quasi-polynomials
Lattice points
Convex bodies
Rational polygons
Scott's inequality

ABSTRACT

In 1976, P. R. Scott characterized the Ehrhart polynomials of convex integral polygons. We study the same question for Ehrhart polynomials and quasi-polynomials of *non*-integral convex polygons. Turning to the case in which the Ehrhart quasi-polynomial has nontrivial quasi-period, we determine the possible minimal periods of the coefficient functions of the Ehrhart quasi-polynomial of a rational polygon.

1. Introduction

A *rational polygon* $P \subset \mathbb{R}^2$ is the convex hull of finitely many rational points, not all contained in a line. Given a positive integer n, let $nP := \{nx \in \mathbb{R}^2 : x \in P\}$ be the dilation of P by n. The 2-dimensional case of a well-known result due to Ehrhart [3] states that the lattice-point enumerator $n \mapsto \left|nP \cap \mathbb{Z}^2\right|$ for P is a degree-2 quasi-polynomial function with rational coefficients. That is, there exist periodic functions $c_{P,0}, c_{P,1}, c_{P,2}\colon \mathbb{Z} \to \mathbb{Q}$ with $c_{P,2} \not\equiv 0$ such that, for all positive integers n,

$$\mathfrak{L}_P(n) := c_{P,0}(n) + c_{P,1}(n)n + c_{P,2}(n)n^2 = \left|nP \cap \mathbb{Z}^2\right|.$$

☆ First author supported by the Netherlands Organisation for Scientific Research (NWO) Mathematics Cluster DIAMANT. Second author supported by Wyoming EPSCoR.

E-mail address: tmcallis@uwyo.edu (T.B. McAllister).

http://dx.doi.org/10.1016/j.jcta.2017.03.002

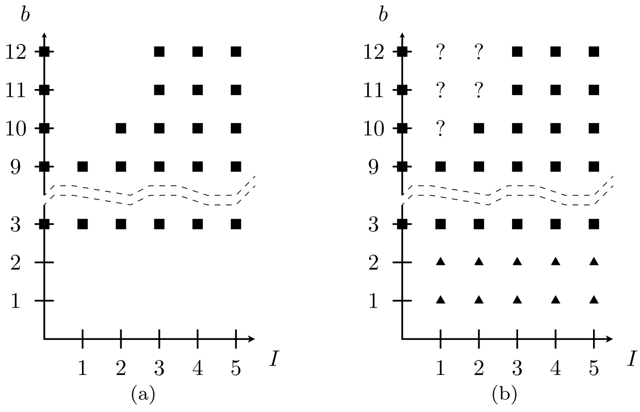

Fig. 1. On the left: Small squares indicate values of (I, b) corresponding to convex integral polygons (Theorem 1.1). On the right: Small triangles indicate additional values of (I, b) corresponding to nonintegral PIPs (Theorem 1.2). Question marks indicate values for which the existence of corresponding PIPs remains open.

We call $\mathfrak{L}_P$ the *Ehrhart quasi-polynomial* of P. The *period sequence* of P is (s_0, s_1, s_2), where s_i is the minimum period of the coefficient function $c_{P,i}$ for $i = 0, 1, 2$. The *quasi-period* of $\mathfrak{L}_P$ (or of P) is $\operatorname{lcm}\{s_0, s_1, s_2\}$. We refer the reader to [1] and [10, Chapter 4] for thorough introductions to the theory of Ehrhart quasi-polynomials.

Our goal is to examine the possible periods and values of the coefficient functions $c_{P,i}$. The leading coefficient function $c_{P,2}$ is always a constant equal to the area $\mathfrak{A}_P$ of P. Furthermore, when P is an *integral* polygon (meaning that its vertices are all in $\mathbb{Z}^2$), $\mathfrak{L}_P$ is simply a polynomial with $c_{P,0} = 1$ and $c_{P,1} = \frac{1}{2}\mathfrak{b}_P$, where $\mathfrak{b}_P$ is the number of lattice points on the boundary of P. When P is integral, Pick's formula $\mathfrak{A}_P = \mathfrak{I}_P + \frac{1}{2}\mathfrak{b}_P - 1$ determines $\mathfrak{A}_P$ in terms of $\mathfrak{b}_P$ and the number $\mathfrak{I}_P$ of points in the interior of P [8]. Hence, characterizing the Ehrhart polynomials of integral polygons amounts to determining the possible numbers of lattice points in their interiors and on their boundaries. This was accomplished by P. R. Scott in 1976:

Theorem 1.1 *(Scott [9]; see also [5]). Given non-negative integers I and b, there exists an integral polygon P such that $(\mathfrak{I}_P, \mathfrak{b}_P) = (I, b)$ if and only if $b \geq 3$ and either $I = 0$, $(I, b) = (1, 9)$, or $b \leq 2I + 6$.*

In Fig. 1, the small squares indicate the values of I and b that are realized as the number of interior lattice points and boundary lattice points of some convex integral polygon. After a suitable linear transformation using Pick's Formula, these squares represent all of the Ehrhart polynomials of integral polygons.

However, not all Ehrhart polynomials of polygons come from *integral* polygons! Indeed, the complete characterization of Ehrhart polynomials of rational polygons, including the nonintegral ones, remains open. To this end, we define a *polygonal pseudo-integral polytope*, or *polygonal PIP*, to be a rational polygon with quasi-period equal to 1. That is, polygonal PIPs are those rational polygons that share with integral polygons the prop-