Journal of Combinatorial Theory, Series A 149 (2017) 101–114

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Gowers' Ramsey Theorem for generalized tetris operations

Martino Lupini

Mathematics Department, California Institute of Technology, 1200 E. California Blvd, MC 253-37, Pasadena, CA 91125, United States

A R T I C L E I N F O

Article history:
Received 6 April 2016
Available online xxxx

Keywords:
Gowers' Ramsey Theorem
Hindman theorem
Milliken–Taylor theorem
Idempotent ultrafilter
Stone–Čech compactification
Partial semigroup

A B S T R A C T

We prove a generalization of Gowers' theorem for FIN_k where, instead of the single tetris operation $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$, one considers all maps from FIN_k to FIN_j for $0 \leq j \leq k$ arising from nondecreasing surjections $f : \{0, 1, \dots, k\} \to \{0, 1, \dots, j\}$. This answers a question of Bartošová and Kwiatkowska. We also describe how to prove a common generalization of such a result and the Galvin–Glazer–Hindman theorem on finite products, in the setting of layered partial semigroups introduced by Farah, Hindman, and McLeod.

1. Introduction

Gowers' theorem on FIN_k is a generalization of Hindman's theorem on finite unions where one considers, rather than finite nonempty subsets of ω, the space FIN_k of all finitely supported functions from ω to $\{0, 1, \dots, k\}$ with maximum value k. Such a space is endowed with a natural operation of pointwise sum, which is defined for pairs of functions

E-mail address: lupini@caltech.edu.
URL: http://www.lupini.org/.

http://dx.doi.org/10.1016/j.jcta.2017.02.001
0097-3165/

with disjoint support. Gowers considered also the *tetris operation* $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ defined by letting $(Tb)(n) = \max\{b(n)-1, 0\}$ for $b \in \mathrm{FIN}_k$. (The term tetris operation has been introduced by Todorcevic [17], and used by Mijares [12], Mijares–Nieto [13], Dobrinen–Mijares [5], Ojeda-Aristizabal [15], and Bartošova–Kwiatkowska [1,2]. It is inspired by the homonymous computer game.) Gowers' theorem can be stated, shortly, by saying that for any finite coloring of FIN_k there exists an infinite sequence (b_n) which is a *block sequence*—in the sense that every element of the support of b_n precedes every element of the support of b_{n+1}—with the property that the intersection of FIN_k with the smallest subset of $\mathrm{FIN}_1 \cup \cdots \cup \mathrm{FIN}_k$ that contains the b_n's and it is closed under pointwise sum of disjointly supported functions and under the tetris operation, is monochromatic [8]. Gowers then used such a result—or more precisely its symmetrized version where one considers functions from ω to $\{-k, \ldots, k\}$—to prove an *oscillation stability* result for the sphere of the Banach space c_0. Other proof of Gowers' theorem can be found in [9,11,17].

Gowers' theorem of FIN_k as stated above implies through a standard compactness argument its corresponding *finitary version*. Explicit combinatorial proofs of such a finitary version have been recently given, independently, by Tyros [18] and Ojeda-Aristizabal [15]. Particularly, the argument from [18] yields a primitive recursive bound on the associated *Gowers' numbers*.

A broad generalization of Gowers' theorem has been proved by Farah, Hindman, and McLeod in [7, Theorem 3.13] in the framework, developed therein, of *layered partial semigroups* and *layered actions*. Such a result provides, in particular, a common generalization of Gowers' theorem and the Hales–Jewett theorem; see [7, Theorem 3.15]. As general as [7, Theorem 3.13] is, it nonetheless does not cover the case where one considers FIN_k endowed with the multiple tetris operations described below, since these do not form a layered action in the sense of [7, Definition 3.3].

In [1], Bartošová and Kwiatkowska considered a generalization of Gowers' theorem, where *multiple tetris operations* are allowed. Precisely, they defined for $1 \leq i \leq k$ the tetris operation $T_{k,i} : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ by

$$T_{k,i}(b) : n \mapsto \begin{cases} b(n) - 1 & \text{if } b(n) \geq i, \text{ and} \\ b(n) & \text{otherwise.} \end{cases}$$

Adapting methods from [18], Bartošová and Kwiatkowska proved in [1] the strengthening of the *finitary version* of Gowers' theorem where multiple tetris operations are considered. The authors then provided in [1] applications of such a result to the dynamics of the Lelek fan.

Question 8.3 of [1] asks whether the *infinitary version* of Gowers' theorem on FIN_k holds when one considers multiple tetris operations. In this paper, we show that this is the case, via an adaptation of Gowers' original argument using idempotent ultrafilters. In order to precisely state our result, we introduce some terminology, to be used in the rest of the paper.