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On the structure of the spectrum of small sets

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ABSTRACT

Let G be a finite Abelian group and A a subset of G. The spectrum of A is the set of its large Fourier coefficients. Known combinatorial results on the structure of spectrum, such as Chang's theorem, become trivial in the regime $|A| = |G|^\alpha$ whenever $\alpha \leq c$, where $c \geq 1/2$ is some absolute constant. On the other hand, there are statistical results, which apply only to a noticeable fraction of the elements, which give nontrivial bounds even to much smaller sets. One such theorem (due to Bourgain) goes as follows. For a noticeable fraction of pairs γ_1, γ_2 in the spectrum, $\gamma_1 + \gamma_2$ belongs to the spectrum of the same set with a smaller threshold. Here we show that this result can be made combinatorial by restricting to a large subset. That is, we show that for any set A there exists a large subset A', such that the sumset of the spectrum of A' has bounded size. Our results apply to sets of size $|A| = |G|^\alpha$ for any constant $\alpha > 0$, and even in some sub-constant regime.

1. Introduction

Let G be a finite Abelian group, and let A be a subset of G. For a character $\gamma \in \widehat{G}$, the corresponding Fourier coefficient of 1_A is

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$$\widehat{1_A}(\gamma) = \sum_{x \in A} \gamma(x).$$

The spectrum of A is the set of characters with large Fourier coefficients,

$$\mathrm{Spec}_\varepsilon(A) = \{\gamma \in \widehat{G} : |\widehat{1_A}(\gamma)| \geq \varepsilon |A|\}.$$

Note that the spectrum of a set is a symmetric set, that is $\mathrm{Spec}_\varepsilon(A) = -\mathrm{Spec}_\varepsilon(A)$, where we view $\widehat{G}$ as an additive group (which is isomorphic to G).

Understanding the structure of the spectrum of sets is an important topic in additive combinatorics, with several striking applications discussed below. As we illustrate, there is a gap in our knowledge between *combinatorial* structural results, which apply to all elements in the spectrum, and *statistical* structural results, which apply to most elements in the spectrum. The former results apply only to large sets, typically of the size $|A| \geq |G|^c$ for some absolute constant $c > 0$, where the latter results apply also for smaller sets. The goal of this paper is to bridge this gap.

Our interest in this problem originates from applications of it in computational complexity, where a better understanding of the structure of the spectrum of small sets can help to shed light on some of the main open problems in the area, such as constructions of two source extractors [4,11,12] or the log rank conjecture in communication complexity [2]. We refer the interested reader to a survey by the second author on applications of additive combinatorics in theoretical computer science [10]. In this paper we focus on the core mathematical problem, and do not discuss applications further.

We assume from now on that $|A| = |G|^\alpha$ where $\alpha > 0$, $\varepsilon > 0$ are arbitrarily small constants, which is the regime where current techniques fail. In fact, our results extend to some range of sub-constant parameters, but only mildly. First, we review the current results on the structure of the spectrum, and their limitations.

Size bound The most basic property of the spectrum is that it cannot be too large. Parseval's identity bounds the size of the spectrum by

$$|\mathrm{Spec}_\varepsilon(A)| \leq \frac{|G|}{\varepsilon^2 |A|} = \frac{|G|^{1-\alpha}}{\varepsilon^2}.$$

However, this does not reveal any information about the structure of the spectrum, except from a bound on its size.

Dimension bound A combinatorial structural result on the spectrum was obtained by Chang [6]. She discovered that the spectrum is low dimensional. For a set $\Gamma \subseteq \widehat{G}$, denote its *dimension* as the minimal integer d, such that there exist $\gamma_1, \ldots, \gamma_d \in \widehat{G}$ with the following property: any element $\gamma \in \Gamma$ can be represented as $\gamma = \sum \varepsilon_i \gamma_i$ with $\varepsilon_i \in \{-1, 0, 1\}$. With this definition, Chang's theorem asserts that

$$\dim(\mathrm{Spec}_\varepsilon(A)) \leq O(\varepsilon^{-2} \log(|G|/|A|)).$$