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Bell numbers, partition moves and the eigenvalues of the random-to-top shuffle in Dynkin Types A, B and D

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ABSTRACT

Let $B_t(n)$ be the number of set partitions of a set of size t into at most n parts and let $B'_t(n)$ be the number of set partitions of $\{1, \ldots, t\}$ into at most n parts such that no part contains both 1 and t or both i and $i+1$ for any $i \in \{1, \ldots, t-1\}$. We give two new combinatorial interpretations of the numbers $B_t(n)$ and $B'_t(n)$ using sequences of random-to-top shuffles, and sequences of box moves on the Young diagrams of partitions. Using these ideas we obtain a very short proof of a generalization of a result of Phatarfod on the eigenvalues of the random-to-top shuffle. We also prove analogous results for random-to-top shuffles that may flip certain cards. The proofs use the Solomon descent algebras of Types A, B and D. We give generating functions and asymptotic results for all the combinatorial quantities studied in this paper.

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1. Introduction

For t, $n \in \mathbf{N}_0$, let $B_t(n)$ be the number of set partitions of $\{1, \ldots, t\}$ into at most n parts. If $n \geq t$ then $B_t(n)$ is the Bell number B_t; the difference $B_t(n) - B_t(n-1)$ is $\left\{ {t \atop n} \right\}$, the Stirling number of the second kind. Let $B'_t(n)$ be the number of set partitions of $\{1, \ldots, t\}$ into at most n parts such that no part contains both 1 and t or both i and $i + 1$ for any $i \in \{1, \ldots, t-1\}$.

The first object of this paper is to give two combinatorial interpretations of the numbers $B_t(n)$ and $B'_t(n)$, one involving certain sequences of random-to-top shuffles, and another involving sequences of box removals and additions on the Young diagrams of partitions. The first of these interpretation is justified by means of an explicit bijection. The second interpretation is considerably deeper, and its justification is less direct: our argument requires the Branching Rule for representations of the symmetric group Sym_n, and a basic result from the theory of Solomon's descent algebra. Using these ideas we obtain a very short proof of a generalization of a result due to Phatarfod [19] on the eigenvalues of the random-to-top shuffle.

We also state and prove analogous results for random-to-top shuffles that may flip the moved card from face-up to face-down, using the descent algebras associated to the Coxeter groups of Type B and D. In doing so, we introduce analogues of the Bell numbers corresponding to these types; these appear not to have been studied previously. We give generating functions, asymptotic formulae and numerical relationships between these numbers, and the associated Stirling numbers, in §6 below.

We now define the quantities which we shall show are equal to either $B_t(n)$ or $B'_t(n)$.

Definition. For $m \in \mathbf{N}$, let σ_m denote the m-cycle $(1, 2, \ldots, m)$. A *random-to-top shuffle* of $\{1, \ldots, n\}$ is one of the n permutations $\sigma_1, \ldots, \sigma_n$. Let $S_t(n)$ be the number of sequences of t random-to-top shuffles whose product is the identity permutation. Define $S'_t(n)$ analogously, excluding the identity permutation σ_1.

We think of the permutations $\sigma_1, \ldots, \sigma_n$ as acting on the n positions in a deck of n cards; thus σ_m is the permutation moving the card in position m to position 1 at the top of the deck. If the cards are labelled by a set C and the card in position m is labelled by $c \in C$ then we say that σ_m *lifts* card c.

We represent partitions by Young diagrams. Motivated by the Branching Rule for irreducible representations of symmetric groups, we say that a box in a Young diagram is *removable* if removing it leaves the Young diagram of a partition; a position to which a box may be added to give a Young diagram of a partition is said to be *addable*.

Definition. A *move* on a partition consists of the removal of a removable box and then addition in an addable position of a single box. A move is *exceptional* if it consists of the removal and then addition in the same place of the lowest removable box. Given partitions λ and μ of the same size, let $M_t(\lambda, \mu)$ be the number of sequences of t moves that start at