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Constructions and nonexistence results for suitable sets of permutations

Justin H.C. Chan, Jonathan Jedwab [1]

Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby BC V5A 1S6, Canada

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ABSTRACT

A set of N permutations of $\{1, 2, \ldots, v\}$ is (N, v, t)-*suitable* if each symbol precedes each subset of $t - 1$ others in at least one permutation. The central problems are to determine the smallest N for which such a set exists for given v and t, and to determine the largest v for which such a set exists for given N and t. These extremal problems were the subject of classical studies by Dushnik in 1950 and Spencer in 1971. We give examples of suitable sets of permutations for new parameter triples (N, v, t). We relate certain suitable sets of permutations with parameter t to others with parameter $t+1$, thereby showing that one of the two infinite families recently presented by Colbourn can be constructed directly from the other. We prove an exact nonexistence result for suitable sets of permutations using elementary combinatorial arguments. We then establish an asymptotic nonexistence result using Ramsey's theorem.

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E-mail addresses: jhc34@sfu.ca (J.H.C. Chan), jed@sfu.ca (J. Jedwab).

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1. Introduction

A set of N permutations of $[v] = \{1, 2, \ldots, v\}$ is (N, v, t)-*suitable* if each symbol precedes each subset of $t - 1$ others in at least one permutation; necessarily we must have $t \leq \min(v, N)$. We represent such a set as an $N \times v$ array called an (N, v, t)-*suitable array*. For example, $\{2413, 3421, 1423\}$ is a $(3, 4, 3)$-suitable set of permutations and its corresponding array is

$$\begin{bmatrix} 2 & 4 & 1 & 3 \\ 3 & 4 & 2 & 1 \\ 1 & 4 & 2 & 3 \end{bmatrix}.$$

Given an (N, v, t)-suitable array, we can readily form an $(N + 1, v, t)$-suitable array by adding an arbitrary extra row, and an $(N, v - 1, t)$-suitable array by removing all occurrences of a single symbol (and left-justifying the remaining symbols). This simple observation motivates two fundamental extremal problems:

(P1) Given v and t, what is the smallest N for which an (N, v, t)-suitable array exists? We denote this as $N(v, t)$ (following [3]), which is well-defined: the $v \times v$ array whose initial elements are $1, 2, \ldots, v$ is (v, v, t)-suitable for each $t \leq v$ and so $N(v, t) \leq v$.

(P2) Given N and t, what is the largest v for which an (N, v, t)-suitable array exists? We denote this as $\mathrm{SUN}(t, N)$ (following [1]). It is well-defined for $t \geq 3$: we then have $\mathrm{SUN}(t, N) \leq 2^{2^N}$ [15], and $\mathrm{SUN}(t, N) \geq N$ by reference to the (v, v, t)-suitable example just described. But $\mathrm{SUN}(2, N)$ is not well-defined for $N \geq 2$, because the $N \times v$ array whose first two rows are $[1 \quad 2 \quad \ldots \quad v-1 \quad v]$ and $[v \quad v-1 \quad \ldots \quad 2 \quad 1]$ is $(N, v, 2)$-suitable for arbitrarily large v.

In 1950, Dushnik [3] introduced problem (P1), showing by combinatorial arguments that $N(v, t) = v - j + 1$ for each j satisfying $2 \leq j \leq \sqrt{v}$ and for each t satisfying

$$\left\lfloor \frac{v}{j} \right\rfloor + j - 1 \leq t < \left\lfloor \frac{v}{j-1} \right\rfloor + j - 2.$$

This determines $N(v, t)$ exactly for all t in the range

$$\left\lfloor \frac{v}{\lfloor \sqrt{v} \rfloor} \right\rfloor + \lfloor \sqrt{v} \rfloor - 1 \leq t < v.$$

In particular, when the lower bound is attained (arising by taking $j = \lfloor \sqrt{v} \rfloor$), both v and $N(v, t)$ grow as $\Theta(t^2)$.

Spencer [15] continued the study of problem (P1) in 1971. Under the condition that $t \geq 3$ is fixed, he used a theorem due to Erdős and Szekeres [6] to show that $N(v, t) \geq \log_2 \log_2 v$ (or equivalently $\mathrm{SUN}(t, N) \leq 2^{2^N}$), and Sperner's lemma [16] and the Erdős–Ko–Rado theorem [5] to show that $N = O(\log_2 \log_2 v)$ as $v \to \infty$.

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