Journal of Combinatorial Theory, Series B ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Skyscraper polytopes and realizations of plane triangulations

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ARTICLE INFO

Article history:
Received 5 November 2013
Available online xxxx

Keywords:
Steinitz
Polytope
Triangulation
Embed
Convex
Plane
Grid
Simplicial
Shedding
Graph

ABSTRACT

We give a new proof of Steinitz's classical theorem in the case of plane triangulations, which allows us to obtain a new general bound on the grid size of the simplicial polytope realizing a given triangulation, subexponential in a number of special cases.

Formally, we prove that every plane triangulation G with n vertices can be embedded in $\mathbb{R}^2$ in such a way that it is the vertical projection of a convex polyhedral surface. We show that the vertices of this surface may be placed in a $4n^3 \times 8n^5 \times \zeta(n)$ integer grid, where $\zeta(n) \leq (500\, n^8)^{\tau(G)}$ and $\tau(G)$ denotes the *shedding diameter* of G, a quantity defined in the paper.

1. Introduction

Steinitz's theorem states every 3-connected plane graph G is the graph of a 3-dimensional convex polytope. An important corollary of the original proof is that the vertices of the polytope can be made integers. The *quantitative Steinitz problem* [20]

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http://dx.doi.org/10.1016/j.jctb.2017.05.007

Please cite this article in press as: I. Pak, S. Wilson, Skyscraper polytopes and realizations of plane triangulations, J. Combin. Theory Ser. B (2017), http://dx.doi.org/10.1016/j.jctb.2017.05.007

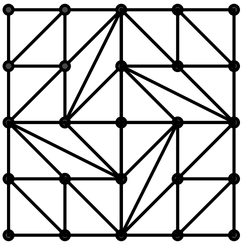

Fig. 1. An example of a grid triangulation of $[5 \times 5]$, with $\ell = 3$.

asks for the smallest size of such integers as they depend on a graph. The best current bounds are exponential in the number of vertices in all three dimensions, even when restricted to triangulations, see [19]. A variant of these bounds, in terms of bit complexity, appears in [7], in which the authors demonstrate that 3-connected planar triangulations can be realized as convex 3-polyhedra whose vertices may be represented using a polynomial number of bits (see [7], Theorem 2.1).

In this paper we improve these bounds in two directions. While the main result of this paper is rather technical (Theorem 4.2), the following corollary requires no background.

Corollary 1.1. *Let G be a plane triangulation with n vertices. Then G is a graph of a convex polyhedron with vertices lying in a $4\,n^3 \times 8\,n^5 \times (500\,n^8)^n$ integer grid.*

This result improves known bounds in two directions at the expense of a somewhat weaker bound in the third direction. We mention that an improvement in one direction, at the expense of the other two, is already given by Schulz, who presents an embedding of general 3-polytopes in an integer grid that is polynomial (in fact linear) in *one* dimension, but superexponential in the other two (see [23], Theorem 3). We call simplicial polytopes obtained from Corollary 1.1 "skyscraper polytopes", as they are small (polynomial in size) in two directions but generally have superexponential size in the third. However, for large families of graphs we make sharp improvements in the third direction as well. Below we give our main application.

A *grid triangulation* of $[a \times b] = \{1, \ldots, a\} \times \{1, \ldots, b\}$ is a triangulation with all grid points as the set of vertices. These triangulations have a curious structure, and have been studied and enumerated in a number of papers (see [1,16,27] and references therein).

Corollary 1.2. *Let G be a grid triangulation of $[k \times k]$, such that every edge sits in an $\ell \times \ell$ subgrid. Then G is a graph of a convex polyhedron with vertices lying in a $O(k^6) \times O(k^{10}) \times k^{O(\ell k)}$ integer grid.*

Setting $\ell = O(1)$ as $k \to \infty$, for the grid triangulations as in the corollary, we have a subexponential grid size in the number $n = k^2$ of vertices: $O(n^3) \times O(n^5) \times \exp O(\sqrt{n}\,\log n)$.