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Fractional clique decompositions of dense graphs and hypergraphs ☆

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Our main result is that every graph G on $n \geq 10^4 r^3$ vertices with minimum degree $\delta(G) \geq (1 - 1/10^4 r^{3/2})n$ has a fractional K_r-decomposition. Combining this result with recent work of Barber, Kühn, Lo and Osthus leads to the best known minimum degree thresholds for exact (non-fractional) F-decompositions for a wide class of graphs F (including large cliques). For general k-uniform hypergraphs, we give a short argument which shows that there exists a constant $c_k > 0$ such that every k-uniform hypergraph G on n vertices with minimum codegree at least $(1 - c_k/r^{2k-1})n$ has a fractional $K_r^{(k)}$-decomposition, where $K_r^{(k)}$ is the complete k-uniform hypergraph on r vertices. (Related fractional decomposition results for triangles have been obtained by Dross and for hypergraph cliques by Dukes as well as Yuster.) All the above new results involve purely combinatorial arguments. In particular, this yields a combinatorial proof of Wilson's

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theorem that every large F-divisible complete graph has an F-decomposition.

1. Introduction and results

1.1. (Fractional) decompositions of graphs

We say that a k-uniform hypergraph G has an *F-decomposition* if its edge set $E(G)$ can be partitioned into copies of F. A natural relaxation is that of a fractional decomposition. To define this, let $\mathcal{F}(G)$ be the set of copies of F in G. A *fractional F-decomposition* is a function $\omega : \mathcal{F}(G) \to [0, 1]$ such that, for each $e \in E(G)$,

$$\sum_{F \in \mathcal{F}(G)\colon\ e \in E(F)} \omega(F) = 1. \tag{1.1}$$

Note that every F-decomposition is a fractional F-decomposition where $\omega(F) \in \{0, 1\}$. As a partial converse, Haxell and Rödl [12] used Szemerédi's regularity lemma to show that the existence of a fractional F-decomposition of a graph G implies the existence of an approximate F-decomposition of G, i.e. a set of edge-disjoint copies of F in G which cover almost all edges of G (their main result is more general than this). Rödl, Schacht, Siggers and Tokushige [17] later generalised this result to k-uniform hypergraphs.

The study of F-decompositions of cliques is central to design theory and has a long and rich history. In 1847, Kirkman [14] showed that K_n has a K_3-decomposition if and only if $n \equiv 1, 3 \mod 6$. More generally, we say that a graph G is *F-divisible* if $e(F)$ divides $e(G)$ and the greatest common divisor of the degrees of F divides the degree of every vertex of G. If G has an F-decomposition then it is certainly F-divisible. Wilson [18–21] proved that if G is a large complete graph, then this necessary condition is also sufficient.

For a given graph F, it is probably not possible to find a satisfactory characterization of all graphs G which have an F-decomposition. This is supported by the fact that Dor and Tarsi [4] proved that determining whether a graph G has an F-decomposition is NP-complete if F has a connected component with at least 3 edges. However, it is natural to ask whether one can extend Kirkman's result and Wilson's theorem to all dense graphs. In particular, Nash-Williams made the following conjecture on triangle decompositions.

Conjecture 1.1 *(Nash-Williams [16]). There exists $N \in \mathbb{N}$ so that for all $n \geq N$, if G is a K_3-divisible graph on n vertices and $\delta(G) \geq 3n/4$, then G has a K_3-decomposition.*

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