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# Classification of reflexible Cayley maps for dihedral groups ☆

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A regular map $\mathcal{M}$ is a 2-cell embedding of a connected graph into an orientable surface such that the group of all orientation-preserving automorphisms of the embedding acts transitively on the set of all incident vertex-edge pairs called arcs. Such a map $\mathcal{M}$ is called a regular Cayley map for the finite group $G$ if $\mathcal{M}$ is the embedding of a Cayley graph $C(G, S)$ such that $G$ induces a vertex-transitive group of map automorphisms preserving orientation. In addition, if there is an orientation-reversing automorphism, the map is called reflexible. In this paper, we classify all reflexible Cayley maps for dihedral groups.



## 1. Introduction

Let $G$ be a finite group, and let $X \subseteq G \backslash \{1_G\}$ be a generating set for $G$ which is closed under taking inverses. The *Cayley graph* $C(G, X)$ for the pair $(G, X)$ has vertex set $G$,

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with any two vertices $g, h \in G$ joined by an edge whenever $g^{-1}h \in X$. It follows that the Cayley graphs considered in this article are finite, connected, undirected and simple.

A (topological) *map* is a 2-cell embedding of a connected graph into a closed surface. We are interested in particular embeddings of Cayley graphs on surfaces. It is well known that to describe an embedding of a graph into an orientable surface, one just needs to specify, at every vertex, a cyclic ordering of edges emanating from the vertex. A graph automorphism which can be extended to a homeomorphism of the supporting surface onto itself is called a *map automorphism*. If the group of orientation-preserving automorphisms of an embedding acts transitively on incident vertex-edge pairs called *arcs*, then this action is also regular. In this case, the embedding (map) is called *regular*. In addition, if there is an orientation-reversing map automorphism, the map is called *reflexible*, and otherwise it is called *chiral*.

For a Cayley graph $C(G, X)$, all edges incident to a vertex $g \in G$ have the form $\{g, gx\}$ where $x \in X$. Hence to describe an embedding of a Cayley graph in an orientable surface, it suffices to specify a cyclic permutation of the generating set at each vertex. If these cyclic permutations are the same at each vertex, which can be viewed as a cyclic permutation $p$ of the set $X$, then the embedding is called a *Cayley map* and is denoted by $CM(G, X, p)$. In this paper, when we describe $p$ clearly, we sometimes omit the description of $X$ because $p$ contains full information about $X$. Since left multiplication by any fixed element of $G$ induces an orientation-preserving automorphism of $CM(G, X, p)$, Cayley maps are vertex-transitive. So a Cayley map $CM(G, X, p)$ is regular if and only if the stabilizer $\mathrm{Aut}(\mathcal{M})_{1_G}$ of the vertex $1_G$ in the group $\mathrm{Aut}(\mathcal{M})$ acts regularly on the arcs emanating from $1_G$. A generator of $\mathrm{Aut}(\mathcal{M})_{1_G}$ satisfies a certain system of identities [5], subsequently re-stated in terms of so-called skew-morphisms [6] which will be introduced later. For more information on regular Cayley maps, the reader is referred to [1,6,12–14].

A Cayley map $\mathcal{M} = CM(G, X, p)$ is called *t-balanced* if $p(x)^{-1} = p^t(x^{-1})$ for every $x \in X$. In particular, if $t = 1$ then $\mathcal{M}$ is called *balanced*, and if $t = -1$, then $\mathcal{M}$ is said to be *anti-balanced*. Note that a Cayley map $\mathcal{M} = CM(G, X, p)$ is regular and balanced if and only if there exists a group automorphism $\psi$ of $G$ whose restriction on $X$ is $p$. Two Cayley maps $CM(G_1, X_1, p_1)$ and $CM(G_2, X_2, p_2)$ are said to be *equivalent* if there exists a group isomorphism $\phi : G_1 \to G_2$ mapping $X_1$ to $X_2$ such that $\phi p_1 = p_2 \phi$. Equivalent Cayley maps are isomorphic as maps. On the other hand, isomorphic Cayley maps may not be equivalent as Cayley maps.

The class of cyclic groups is the only class of finite groups for which all regular Cayley maps have been classified due to the work of Conder and Tucker [4]. Regarding other groups, only partial classifications are known (see, e.g. [9–11,15]). In particular, considerable work has been devoted to regular Cayley maps for dihedral groups. In this paper, the dihedral group of order $2n$, $n \geq 2$, will be denoted by $D_n$. A complete classification of regular Cayley maps for $D_n$ has been given in [15] for balanced maps; in [11] for $t$-balanced maps with $t > 1$; in [9] for non-balanced maps with odd $n$; and in [17] for maps of so called skew-type 3. Denoted by $\varphi$ the skew-morphism of $\mathcal{M}$ and by $\pi$ the associated power function (for the definition of $\varphi$ and $\pi$, see Section 2), the

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