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Breaking graph symmetries by edge colourings

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ABSTRACT

The distinguishing index $D'(G)$ of a graph G is the least number of colours needed in an edge colouring which is not preserved by any non-trivial automorphism. Broere and Pilśniak conjectured that if every non-trivial automorphism of a countable graph G moves infinitely many edges, then $D'(G) \leq 2$. We prove this conjecture.

1. Introduction

A colouring of the vertices or edges of a graph G is called distinguishing if the only automorphism which preserves it is the identity. Originally inspired by a recreational mathematics problem, Albertson and Collins [1] first introduced the notion formally in 1996. The concept quickly received a lot of attention despite (or maybe because of) its recreational origin, leading to numerous papers on distinguishing colourings of graphs and other combinatorial structures.

One interesting line of research involves the motion $m(G)$ of a graph which is defined as the minimal number of vertices moved by a non-trivial automorphism. Intuitively, the larger the motion of a graph is, the easier it should be to find a distinguishing vertex colouring with few colours. Russel and Sundaram [13] were the first to make this intuition precise. They showed that if $|\operatorname{Aut} G| \leq 2^{\frac{m(G)}{2}} < \infty$, then there is a distinguishing vertex

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2-colouring. Tucker [15] conjectured, that an analogous result holds for locally finite graphs with infinite automorphism group.

Conjecture 1 *(Infinite motion conjecture [15]). Let G be a locally finite, connected graph and assume that every automorphism of G moves infinitely many vertices. Then there is a distinguishing vertex 2-colouring.*

Let us briefly check that this is indeed a generalisation of the result of Russel and Sundaram. Recall that the vertex set of a locally finite graph is always countable. Clearly the motion of a graph is always bounded from above by the number of vertices. So in case the motion is infinite we get $|V| = m(G)$. Furthermore note that the automorphism group of any graph contains at most $|V|^{|V|}$ many elements. Finally recall that if α is an infinite cardinal, then $\frac{\alpha}{2} = \alpha$ and $\alpha^\alpha = 2^\alpha$. Together these observations give that

$$|\operatorname{Aut} G| \leq |V|^{|V|} = 2^{|V|} \leq 2^{m(G)} \leq 2^{\frac{m(G)}{2}},$$

hence the condition $|\operatorname{Aut} G| \leq 2^{\frac{m(G)}{2}}$ is satisfied.

While Tucker's conjecture is still wide open, there are many partial results towards it, see for example [3,6,7,10,11,14,16]. Broere and Pilśniak [2] noticed that most of the partial results towards Conjecture 1 can be generalised to edge colourings. Consequently, they conjectured that an analogous statement should hold in the realm of edge colourings. In fact their conjecture for edge colourings is even stronger as it does not require the graph to be locally finite.

Conjecture 2 *(Infinite edge motion conjecture [2]). Let G be a countable, connected graph and assume that every automorphism of G moves infinitely many edges. Then there is a distinguishing edge 2-colouring.*

The two conjectures are closely related. In [8] the following generalisation of Whitney's Theorem is proved. Let G be a connected graph on more than 4 vertices. Then there is a natural isomorphism between $\operatorname{Aut} G$ and $\operatorname{Aut} L(G)$, where $L(G)$ denotes the line graph of G. In particular, this holds for all infinite graphs. Hence, a distinguishing vertex colouring of $L(G)$ translates into a distinguishing edge colouring of G and vice versa. So Conjecture 1 implies the special case of Conjecture 2 where the graph is assumed to be locally finite.

If the generalisation of Conjecture 1 to countable graphs were true, then this would immediately imply Conjecture 2. However, in [12] a counterexample for this generalisation is constructed, making it somewhat counterintuitive that Conjecture 2 holds in full generality.

Nevertheless, in the present paper we prove Conjecture 2. We also attempt to give some intuition why this is not as surprising as it may seem at first glance. For this purpose, in Section 4 we compare distinguishing edge and vertex colourings. We show that if there

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