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Robust Hamiltonicity of random directed graphs

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A B S T R A C T

In his seminal paper from 1952 Dirac showed that the complete graph on $n \geq 3$ vertices remains Hamiltonian even if we allow an adversary to remove $\lfloor n/2 \rfloor$ edges touching each vertex. In 1960 Ghouila–Houri obtained an analogue statement for digraphs by showing that every directed graph on $n \geq 3$ vertices with minimum in- and out-degree at least $n/2$ contains a directed Hamilton cycle. Both statements quantify the robustness of complete graphs (digraphs) with respect to the property of containing a Hamilton cycle.

A natural way to generalize such results to arbitrary graphs (digraphs) is using the notion of *local resilience*. The local resilience of a graph (digraph) G with respect to a property $\mathcal{P}$ is the maximum number r such that G has the property $\mathcal{P}$ even if we allow an adversary to remove an r-fraction of (in- and out-going) edges touching each vertex. The theorems of Dirac and Ghouila–Houri state that the local resilience of the complete graph and digraph with respect to Hamiltonicity is 1/2. Recently, this statements have been generalized to random settings. Lee and Sudakov (2012) proved that the local resilience of a random graph with edge probability $p = \omega(\log n/n)$ with respect to Hamiltonicity is $1/2 \pm o(1)$. For random directed graphs, Hefetz, Steger and Sudakov (2014) proved an analogue statement, but only for edge probability $p = \omega(\log n/\sqrt{n})$. In this paper we significantly improve their

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result to $p = \omega\left(\log^8 n/n\right)$, which is optimal up to the polylogarithmic factor.

1. Introduction

A *Hamilton cycle* in a graph or a directed graph is a cycle that passes through all the vertices of the graph exactly once, and a graph is *Hamiltonian* if it contains a Hamilton cycle. Hamiltonicity is one of the central notions in graph theory, and has been intensively studied by numerous researchers. It is well-known that the problem of whether a given graph contains a Hamilton cycle is $\mathcal{NP}$-complete. In fact, Hamiltonicity was one of Karp's 21 $\mathcal{NP}$-complete problems [12].

Since one can not hope for a general classification of Hamiltonian graphs, as a consequence of Karp's result, there is a large interest in deriving properties that are sufficient for Hamiltonicity. A classic result by Dirac from 1952 [7] states that every graph on $n \geq 3$ vertices with minimum degree at least $n/2$ is Hamiltonian. This result is tight as the complete bipartite graph with parts of sizes that differ by one, $K_{m,m+1}$, is not Hamiltonian. Note that it also answers the following question: Starting with the complete graph on n vertices K_n, what is the maximal integer Δ such that for any subgraph H of K_n with maximum degree Δ, the graph $K_n - H$ obtained by deleting the edges of H from K_n is Hamiltonian? This question not only asks for a sufficient condition for a graph to be Hamiltonian, it also asks for a quantification for the "local robustness" of the complete graph with respect to Hamiltonicity.

A natural generalization of this question is to replace the complete graph with some other base graph. Recently, questions of this type have drawn a lot of attention under the notion of *resilience*.

Roughly speaking, given a monotone increasing graph property $\mathcal{P}$ and a graph or a digraph G which satisfies $\mathcal{P}$, the *resilience of G with respect to $\mathcal{P}$* measures how much one must change G in order to destroy $\mathcal{P}$. Since one can destroy many natural properties by small changes (for example, by isolating a vertex), it is natural to limit the number of edges touching any vertex that one is allowed to delete. This leads to the following definition of *local resilience*.

Definition 1.1 *(Local resilience).* Let $\mathcal{P}$ be a monotone increasing graph property. For a graph G, the local resilience is

$$\begin{aligned} r(G,\mathcal{P}) := \min\{r : \exists H \subseteq G \text{ such that } &\forall v \in V(G) \\ &d_H(v) \leq r \cdot d_G(v) \text{ and} \\ &G - H \text{ does not have } \mathcal{P}\}, \end{aligned}$$