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Maximising the number of induced cycles in a graph ☆

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A R T I C L E I N F O

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We determine the maximum number of induced cycles that can be contained in a graph on $n \geq n_0$ vertices, and show that there is a unique graph that achieves this maximum. This answers a question of Chvátal and Tuza from the 1980s. We also determine the maximum number of odd or even induced cycles that can be contained in a graph on $n \geq n_0$ vertices and characterise the extremal graphs. This resolves a conjecture of Chvátal and Tuza from 1988.

1. Introduction

What is the maximum number of induced cycles in a graph on n vertices? For cycles of *fixed* length, this problem has been extensively studied. Indeed, for any fixed graph H, let the *induced density* of H in a graph G be the number of induced copies of H in G divided by $\binom{|G|}{|H|}$; let $I(H;n)$ be the maximum induced density of H over all graphs G on n vertices; and let the *inducibility* of H be the limit $\lim_{n\to\infty} I(H;n)$. In 1975, Pippinger and Golumbic [12] made the following conjecture.

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Conjecture 1.1. [12] *For $k \geq 5$, the inducibility of the cycle C_k is $k!/(k^k - k)$.*

Balogh, Hu, Lidický and Pfender [2] recently proved this conjecture in the case $k = 5$ via a flag algebra method, and showed that the maximum density was achieved by a unique graph. Apart from this case, the problem remains open (though see [3–9] for results on inducibility of other graphs).

In this paper, we consider the total number of induced cycles, without restriction on length. This problem was raised in the 1980s by Chvátal and Tuza (see [15] and [16]), who asked for the maximum possible number of induced cycles in a graph with n vertices. The problem was investigated independently in unpublished work of Robson, who showed in the 1980s that a graph on n vertices has at most $3^{(1+o(1))n/3}$ induced cycles ([10,13]). Tuza also raised a second, closely related problem on induced cycles. In 1988 he conjectured with Chvátal (see [14,15] and [16]) that the maximum possible number of odd induced cycles in a graph on n vertices is $3^{n/3}$.

In this paper we resolve both problems, proving exact bounds for all sufficiently large n, and determining the extremal graphs. Our methods work for a number of problems of this type: thus we will determine, for sufficiently large n, the graphs with n vertices that maximize the number of induced cycles (Theorem 1.2); we will also determine the graphs with the maximum number of even induced cycles, the maximum number of odd induced cycles (Theorem 1.6), and in Theorem 1.7, the maximum number of odd holes (i.e. induced odd cycles of length at least 5).

In order to state our results, it is helpful to have a couple of definitions. As usual, for G a graph define the *neighbourhood* of x to be $N_G(x) := \{y \in V(G) : xy \in E(G)\}$. A graph B is called a *cyclic braid* if there exists $k \geq 3$ and a partition $B_1, \dots, B_k$ of $V(B)$ such that for every $1 \leq i \leq k$ and every $x \in B_i$, we have $B_{i-1} \cup B_{i+1} \subseteq N_B(x) \subseteq B_{i-1} \cup B_i \cup B_{i+1}$ where indices are taken modulo k. For such a partition, the notation $B = (B_1, \dots, B_k)$ is used. The sets B_i are called *clusters* of B; the *length* of the cyclic braid is the number of clusters. If a cyclic braid contains no edges within its clusters, it is called an *empty cyclic braid*. Observe that an empty cyclic braid is k-partite and, when $k > 3$, is triangle free. If a cyclic braid contains every possible edge within each cluster, then it is called *full*. A pair of clusters B_1 and B_2 are *adjacent* in G if $v_1v_2 \in E(G)$ for all $v_1 \in B_1$ and $v_2 \in B_2$. A triple of clusters B_1, B_2, B_3 are *consecutive* if B_1 is adjacent to B_2 and B_2 is adjacent to B_3.

As it turns out, the structure of the extremal graph depends on the value of n modulo 3. For $n \geq 8$ define an n-vertex graph H_n separately for each value of n modulo 3. Let $k \geq 3$. Define H_{3k} to be the empty cyclic braid of length k where every cluster has size 3. Define H_{3k+1} to be the empty cyclic braid containing $k-1$ clusters of size 3 and one of size 4. Finally, define H_{3k-1} to be the empty cyclic braid containing $k-1$ clusters of size 3 and one of size 2.

Let $m(n)$ be the maximum number of induced cycles that can be contained in a graph on n vertices. The main result of our paper is the following.