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Tournaments, 4-uniform hypergraphs, and an exact extremal result ☆

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ABSTRACT

We consider 4-uniform hypergraphs with the maximum number of hyperedges subject to the condition that every set of 5 vertices spans either 0 or exactly 2 hyperedges and give a construction, using quadratic residues, for an infinite family of such hypergraphs with the maximum number of hyperedges. Baber has previously given an asymptotically best-possible result using random tournaments. We give a connection between Baber's result and our construction via Paley tournaments and investigate a 'switching' operation on tournaments that preserves hypergraphs arising from this construction.

1. Introduction

For any $r \geq 2$, an r-uniform hypergraph $\mathcal{H}$, and integer n, the *Turán number* for $\mathcal{H}$, denoted $\text{ex}(\mathcal{H}, n)$, is the maximum number of hyperedges in any r-uniform hypergraph on n vertices containing no copy of $\mathcal{H}$. The *Turán density* of $\mathcal{H}$ is defined to be $\pi(\mathcal{H}) =$

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$\lim_{n\to\infty} \mathrm{ex}(\mathcal{H}, n)\binom{n}{r}^{-1}$ and a well-known averaging argument shows that this limit always exists. While the Turán densities of graphs are well-understood and exact Turán numbers are known for some classes of graphs, few exact results for either Turán numbers or densities are known for the cases $r \geq 3$. The interested reader is directed to the excellent survey by Keevash [29] for more details on known Turán results for hypergraphs.

One particular extremal problem, which has a close connection to the Turán density of a particular 3-uniform hypergraph was considered by Frankl and Füredi [22]. Recall that a subset of vertices, X, in a hypergraph is said to span an edge A if $A \subseteq X$. In their paper, Frankl and Füredi considered those 3-uniform hypergraphs that have the property that any set of 4 vertices span either 0 or exactly 2 hyperedges. They proved the exact and strongly structural result that any such hypergraph is in one of two classes: either it is the blow-up of a fixed 3-graph on 6 vertices with 10 edges, or else it is isomorphic to a hypergraph obtained by taking vertices as points around a unit circle and then letting the hyperedges be those triples whose convex hull contains the origin (assuming that no two points lie on a line containing the origin).

This result by Frankl and Füredi has an application to the Turán density of a small 3-uniform hypergraph. Let K_4^- be the 3-uniform hypergraph on 4 vertices with 3 edges. Using a recursive construction based on the 3-uniform hypergraph on 6 vertices mentioned above, Frankl and Füredi show that $\pi(K_4^-) \geq \frac{2}{7}$. Using flag algebra techniques, Baber and Talbot [5] showed that the Turán density for K_4^- is at most 0.2871. It is conjectured in this case that the exact value is 2/7 (*e.g.* [33]). Many further results on extremal numbers for small cliques in hypergraphs can be found, for example, in [6,18,21,23,30,31,35,37,39]. Further specialized versions of these extremal problems, in which hypergraphs have a 'uniform edge density', known as δ-linear density were proposed by Erdős and Sós [20] with recent results for this type of question on K_4^- by Glebov, Král', and Volec [26] also using the method of flag algebras.

In the same paper, Frankl and Füredi ask about the following more general question. For any $r \geq 4$, what is the maximum number of hyperedges in an r-uniform hypergraph with the property that any set of $r+1$ vertices span 0 or 2 edges? They point out that the construction given by points on a circle can be generalized to points on a sphere in $r-1$ dimensions with hyperedges being simplices containing the origin. Frankl and Füredi note that such an r-uniform hypergraph has at most $\binom{n}{r}2^{-r+1}(1 + o(1))$ hyperedges for n tending to infinity and that a random choice of vertices on the sphere gives this number of hyperedges.

In this paper, we consider their question in the case $r = 4$ and give a construction for an infinite family of 4-uniform hypergraphs with the property that every set of 5 vertices spans either 0 or 2 hyperedges with the maximum number of hyperedges among all such hypergraphs on the same number of vertices. One of the main results of this paper is Theorem 1 below, whose proof appears in Section 2.

Before stating this result, we introduce and recall some terminology. An action of a group G on a set V is 3-*transitive* if given two 3-tuples (x, y, z) and (x', y', z') of elements of V there exists $g \in G$ such that $(xg, yg, zg) = (x', y', z')$. A hypergraph is defined to be