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Three conjectures in extremal spectral graph theory

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A B S T R A C T

We prove three conjectures regarding the maximization of spectral invariants over certain families of graphs. Our most difficult result is that the join of P_2 and P_{n-2} is the unique graph of maximum spectral radius over all planar graphs. This was conjectured by Boots and Royle in 1991 and independently by Cao and Vince in 1993. Similarly, we prove a conjecture of Cvetković and Rowlinson from 1990 stating that the unique outerplanar graph of maximum spectral radius is the join of a vertex and P_{n-1}. Finally, we prove a conjecture of Aouchiche et al. from 2008 stating that a pineapple graph is the unique connected graph maximizing the spectral radius minus the average degree. To prove our theorems, we use the leading eigenvector of a purported extremal graph to deduce structural properties about that graph.

1. Introduction

Questions in *extremal graph theory* ask to maximize or minimize a graph invariant over a fixed family of graphs. Perhaps the most well-studied problems in this area are

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Turán-type problems, which ask to maximize the number of edges in a graph which does not contain fixed forbidden subgraphs. Over a century old, a quintessential example of this kind of result is Mantel's theorem, which states that $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ is the unique graph maximizing the number of edges over all triangle-free graphs. *Spectral graph theory* seeks to associate a matrix to a graph and determine graph properties by the eigenvalues and eigenvectors of that matrix. This paper studies the maximization of spectral invariants over various families of graphs. We prove three conjectures for n large enough.

Conjecture 1 *(Boots–Royle 1991 [8] and independently Cao–Vince 1993 [10]). The planar graph on $n \geq 9$ vertices of maximum spectral radius is $P_2 + P_{n-2}$.*

Conjecture 2 *(Cvetković–Rowlinson 1990 [13]). The outerplanar graph on n vertices of maximum spectral radius is $K_1 + P_{n-1}$.*

Conjecture 3 *(Aouchiche et al. 2008 [3]). The connected graph on n vertices that maximizes the spectral radius minus the average degree is a pineapple graph.*

In this paper, we prove Conjectures 1, 2, and 3, with the caveat that we must assume n is large enough in all of our proofs. We note that the Boots–Royle/Vince–Cao conjecture is not true when $n \in \{7, 8\}$ and thus some bound on n is necessary.

For each theorem, the rough structure of our proof is as follows. A lower bound on the invariant of interest is given by the conjectured extremal example. Using this information, we deduce the approximate structure of a (planar, outerplanar, or connected) graph maximizing this invariant. We then use the leading eigenvalue and eigenvector of the adjacency matrix of the graph to deduce structural properties of the extremal graph. Once we know the extremal graph is "close" to the conjectured graph, we show that it must be exactly the conjectured graph. The majority of the work in each proof is done in the step of using the leading eigenvalue and eigenvector to deduce structural properties of the extremal graph.

1.1. History and motivation

Questions in extremal graph theory ask to maximize or minimize a graph invariant over a fixed family of graphs. This question is deliberately broad, and as such branches into several areas of mathematics. We already mentioned Mantel's Theorem as an example of a theorem in extremal graph theory. Other classic examples include the following. Turán's Theorem [35] seeks to maximize the number of edges over all n-vertex K_r-free graphs. The Four Color Theorem seeks to maximize the chromatic number over the family of planar graphs. Questions about maximum cuts over various families of graphs have been studied extensively (cf. [2,7,11,18]). The Erdős distinct distance problem seeks to minimize the number of distinct distances between n points in the plane [16,20].

This paper studies *spectral extremal graph theory*, the subset of these extremal problems where invariants are based on the eigenvalues or eigenvectors of a graph. This subset