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The extremal function for disconnected minors

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ABSTRACT

For a graph H let $c(H)$ denote the supremum of $|E(G)|/|V(G)|$ taken over all non-null graphs G not containing H as a minor. We show that

$$c(H) \leq \frac{|V(H)| + \text{comp}(H)}{2} - 1,$$

when H is a union of cycles. This verifies a conjecture of Reed and Wood, and another conjecture of Harvey and Wood. We derive the above result from a theorem which allows us to find two vertex-disjoint subgraphs with prescribed densities in a sufficiently dense graph, which might be of independent interest.

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1. Introduction

A classical theorem of Erdős and Gallai determines the minimum number of edges necessary to guarantee existence of a cycle of length at least k in a graph with a given number of vertices. (All the graphs considered in this paper are simple.)

Theorem 1 *(Erdős and Gallai [3]). Let $k \geq 3$ be an integer and let G be a graph with n vertices and more than $(k-1)(n-1)/2$ edges. Then G contains a cycle of length at least k.*

One of the main results of this paper generalizes Theorem 1 to a setting where, instead of a single cycle with prescribed minimum length, we are interested in obtaining a collection of vertex-disjoint cycles. In the case when there are no restrictions on the lengths of cycles this problem was completely solved by Dirac and Justesen, who proved the following.

Theorem 2 *(Dirac and Justesen [7]). Let $k \geq 2$ be an integer and let G be a graph with $n \geq 3k$ vertices and more than*

$$\max\left\{(2k-1)(n-k), n - \frac{(3k-1)(3k-4)}{2}\right\}$$

edges. Then G contains k vertex-disjoint cycles.

We phrase our extensions of the above results in the language of minors. A graph H is *a minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. Mader [14] proved that for every graph H there exists a constant c such that every graph on $n \geq 1$ vertices with at least cn edges contains H as a minor. A well-studied extremal question in graph minor theory is determining the optimal value of c for a given graph H. Denote by $v(G)$ and $e(G)$ the number of vertices and edges of a graph G, respectively. Following Myers and Thomason [16], for a graph H with $v(H) \geq 2$ we define $c(H)$ as the supremum of $e(G)/v(G)$ taken over all non-null graphs G not containing H as a minor. We refer to $c(H)$ as *the extremal function of H*.

The extremal function of complete graphs has been extensively studied. Dirac [2], Mader [14], Jørgensen [6], and Song and Thomas [19] proved that $c(K_t) = t-2$ for $t \leq 5$, $t \leq 7$, $t = 8$ and $t = 9$, respectively. The asymptotic behaviour of $c(K_t)$ for large t was studied in [9,10,21], and was determined precisely by Thomason [22], who has shown that

$$c(K_t) = (\alpha + o_t(1))t\sqrt{\log t},$$

for an explicit constant $\alpha = 0.37...$. Myers and Thomason [16] have extended the results of [22] to general dense graphs, while Reed and Wood [18] and Harvey and Wood [5] have