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Every finite non-solvable group admits an oriented regular representation

Joy Morris [a], Pablo Spiga [b]

[a] *Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, AB, T1K 3M4, Canada*
[b] *Dipartimento di Matematica Pura e Applicata, University of Milano-Bicocca, Via Cozzi 55, 20126 Milano, Italy*

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ABSTRACT

In this paper we give a partial answer to a 1980 question of Lazslo Babai: "Which [finite] groups admit an oriented graph as a DRR?" That is, which finite groups admit an oriented regular representation (ORR)? We show that every finite non-solvable group admits an ORR, and provide a tool that may prove useful in showing that some families of finite solvable groups admit ORRs. We also completely characterize all finite groups that can be generated by at most three elements, according to whether or not they admit ORRs.

1. Introduction

All groups and graphs in this paper are finite. Let G be a group and let S be a subset of G. The *Cayley digraph* $\mathrm{Cay}(G,S)$ over G with connection set S is the digraph with vertex set G and with (x,y) being an arc if $yx^{-1} \in S$. (In this paper, an *arc* is an ordered pair of adjacent vertices.) It is easy to see that the group G acts faithfully as a group of automorphisms of $\mathrm{Cay}(G,S)$ via the right regular representation. In particular,

E-mail addresses: joy.morris@uleth.ca (J. Morris), pablo.spiga@unimib.it (P. Spiga).

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Cayley digraphs offer a natural way to represent groups geometrically and combinatorially as groups of automorphisms of digraphs. Clearly, this representation is particularly meaningful if G is the full automorphism group of $\mathrm{Cay}(G, S)$.

In this context it is fairly natural to ask which groups G admit a subset S with G being the automorphism group of $\mathrm{Cay}(G, S)$; that is, $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. In this case, we say that G admits a *digraphical regular representation* (or DRR for short). Babai [1, Theorem 2.1] has given a complete classification of the groups admitting a DRR: except for

$$Q_8,\ C_2^2,\ C_2^3,\ C_2^4 \text{ and } C_3^2,$$

every group admits a DRR. (Throughout this paper, Q_8 denotes the quaternion group of order 8.)

In light of Babai's result, it is natural to try to combinatorially represent groups as automorphism groups of special classes of Cayley digraphs. Observe that if S is inverse-closed (that is, $S = \{s^{-1} \mid s \in S\} := S^{-1}$), then $\mathrm{Cay}(G, S)$ is undirected. Now, we say that G admits a *graphical regular representation* (or GRR for short) if there exists an inverse-closed subset S of G with $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. With a considerable amount of work culminating in [9,10], the groups admitting a GRR have been completely classified. (The pioneer work of Imrich [11–13] was an important step towards this classification.) It is interesting to observe that, although the classification of the groups admitting a DRR is much easier than the classification of the groups admitting a GRR, research and interest first focused on finding GRRs and then on DRRs. It is also worth noting that various researchers have shown that for certain families of groups, almost all Cayley graphs are GRRs, or almost all Cayley digraphs are DRRs [2,6,7,9].

We recall that a *tournament* is a digraph $\Gamma = (V, A)$ with vertex set V and arc set A such that, for every two distinct vertices $x, y \in V$, exactly one of (x, y) and (y, x) is in A. After the completion of the classification of DRRs and GRRs, Babai and Imrich [3] proved that every group of odd order except $C_3 \times C_3$ admits a *tournament regular representation* (or TRR for short). That is, each of these groups G admits a subset S with $\mathrm{Cay}(G, S)$ being a tournament and with $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$. In terms of the connection set S, the Cayley digraph $\mathrm{Cay}(G, S)$ is a tournament if and only if $S \cap S^{-1} = \emptyset$ and $G \setminus \{1\} = S \cup S^{-1}$. This observation makes it clear that a Cayley digraph on G cannot be a tournament if G contains an element of order 2, so only groups of odd order can admit TRRs.

In [1, Problem 2.7], Babai observed that there is one class of Cayley digraphs that is rather interesting and that has not been investigated in the context of regular representations; that is, the class of oriented Cayley digraphs (or as Babai called them, oriented Cayley graphs). An *oriented Cayley digraph* is in some sense a "proper" digraph. More formally, it is a Cayley digraph $\mathrm{Cay}(G, S)$ whose connection set S has the property that $S \cap S^{-1} = \emptyset$. Equivalently, in graph-theoretic terms, it is a digraph with no digons.