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An algorithmic framework for obtaining lower bounds for random Ramsey problems

Rajko Nenadov [a], Yury Person [b], Nemanja Škorić [a], Angelika Steger [a]

[a] Department of Computer Science, ETH Zurich, 8092 Zurich, Switzerland
[b] Institute of Mathematics, Goethe-Universität, 60325 Frankfurt am Main, Germany

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ABSTRACT

In this paper we introduce a general framework for proving lower bounds for various Ramsey type problems within random settings. The main idea is to view the problem from an algorithmic perspective: we aim at providing an algorithm that finds the desired coloring with high probability. Our framework allows to reduce the probabilistic problem of whether the Ramsey property at hand holds for random (hyper)graphs with edge probability p to a deterministic question of whether there exists a finite graph that forms an obstruction.

In the second part of the paper we apply this framework to address and solve various open problems. In particular, we extend the result of Bohman, Frieze, Pikhurko and Smyth (2010) for bounded anti-Ramsey problems in random graphs to the case of 2 colors and to hypergraph cliques. As a corollary, this proves a matching lower bound for the result of Friedgut, Rödl and Schacht (2010) and, independently, Conlon and Gowers (2016) for the classical Ramsey problem for hypergraphs in the case of cliques. Finally, we provide matching lower bounds for a proper-coloring version of anti-

E-mail addresses: rajko.nenadov@monash.edu (R. Nenadov), person@math.uni-frankfurt.de (Y. Person), nskoric@inf.ethz.ch (N. Škorić), steger@inf.ethz.ch (A. Steger).

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Ramsey problems introduced by Kohayakawa, Konstadinidis and Mota (2014) in the case of cliques and cycles.

1. Introduction and results

A hypergraph G is *Ramsey* for a hypergraph F and an integer r, if every coloring of the edges of G with r colors contains a copy of F with all its edges having the same color. A celebrated theorem of Ramsey [18] states that if G is a large enough complete hypergraph then G is Ramsey for F and r. A priori it is not clear whether this follows from the density of a complete hypergraph or its rich structure. It was shown only later that actually the latter is the case: there exist sparse graphs with rich enough structure so that they are Ramsey for F. For example, a result of Nešetřil and Rödl [17] states that for every k there exists a graph G that does not contain a clique of size $k+1$, but that nevertheless is Ramsey for a clique of size k. Nowadays, the easiest way to prove such result is by studying Ramsey properties of random (hyper)graphs.

Over the last decades the study of various Ramsey-type problems for random (hyper)graphs received a lot of attention. In their landmark result, Rödl and Ruciński [19–21] gave a precise characterization of all edge probabilities $p = p(n)$ for which Ramsey's theorem holds in the random graph $G(n,p)$ for a given graph F and r colors. The corresponding problem for hypergraphs remained open for more than 15 years. Only recently, Friedgut, Rödl and Schacht [7] and independently Conlon and Gowers [4] obtained an upper bound on the threshold for this property analogous to the graph case. However, the question whether there exists a matching lower bound remained open.

More recently, other variations on Ramsey-type problems in random graphs have been investigated. These are so-called anti-Ramsey properties such as finding rainbow copies of a given graph F in any r-bounded coloring of $G(n,p)$, initiated by Bohman, Frieze, Pikhurko and Smyth [1], and in any proper edge-coloring of $G(n,p)$, introduced by Kohayakawa, Konstadinidis and Mota [11].

The aim of our paper is twofold. First we introduce a general framework for proving lower bounds on thresholds for Ramsey-type problems for random hypergraphs. Roughly speaking, the framework allows to reduce the probabilistic problem

Does the Ramsey property at hand hold for

random (hyper)graphs with edge probability p w.h.p.?

to a deterministic question of whether there exists a (hyper)graph that forms an *obstruction*, or more precisely

Does there exist a (hyper)graph with density at most $d(F,r)$ on at most $v(F,r)$

vertices that does not have the given Ramsey property?