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Multicolour Ramsey numbers of odd cycles

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ABSTRACT

We show that for any positive integer r there exists an integer k and a k-colouring of the edges of K_{2^k+1} with no monochromatic odd cycle of length less than r. This makes progress on a problem of Erdős and Graham and answers a question of Chung. We use these colourings to give new lower bounds on the k-colour Ramsey number of the odd cycle and prove that, for all odd r and all k sufficiently large, there exists a constant $\epsilon = \epsilon(r) > 0$ such that $R_k(C_r) > (r-1)(2+\epsilon)^{k-1}$.

1. Introduction

In this paper all *colourings* of a graph G will refer to colourings of the edges of G. The *odd girth* of G, written $\mathrm{og}(G)$, is the length of the shortest odd cycle in G. Given a colouring $\mathcal{C}$ of G we say the odd girth of $\mathcal{C}$, written $\mathrm{og}(\mathcal{C})$, is the length of the shortest monochromatic odd cycle found in $\mathcal{C}$. It is a simple exercise to see that it is possible to k-colour the complete graph K_{2^k} such that each colour comprises a bipartite graph. Moreover, such colourings only exist for K_n if $n \leqslant 2^k$. Indeed, consider labelling each vertex of K_n with a binary vector of length k, where the ith coordinate of the label given to a vertex is determined by which side of the bipartition of colour i the vertex lies in. All

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vertices of K_n must receive distinct labels, and so $n \leqslant 2^k$. It follows that any k-colouring of K_{2^k+1} must contain a monochromatic odd cycle. Based on this observation, Erdős and Graham [4] asked the following question:

Question 1. *How large can the smallest monochromatic odd cycle in a k-colouring of K_{2^k+1} be?*

Moreover, Chung [3] asked further whether or not this quantity is unbounded as k increases. In this paper we show that the size of the least odd cycle that must appear is indeed unbounded.

Theorem 2. *For all positive integers r there exists an integer k and a k-colouring of K_{2^k+1} with odd girth at least r.*

The proof of Theorem 2 can be found in Section 2. From a quantitative perspective, our proof of Theorem 2 will show that there exist k-colourings of K_{2^k+1} with odd girth at least $2^{\sqrt{2\log_2(k)-c}}$ for some constant c. This result is a consequence of Corollary 6 which can be found at the end of Section 2.

For a graph H, the k-colour Ramsey number $R_k(H)$ is defined as the least integer n such that every k-colouring of K_n contains a monochromatic copy of H. We say that a colouring of a graph G is *H-free* if it contains no monochromatic copy of H. Erdős and Graham [4] showed that

$$R_k(C_r) \geqslant (r-1)2^{k-1} + 1 \tag{1}$$

whenever $r \geqslant 3$ is an odd integer. The construction used to show this is as follows: When $k = 1$ simply take a 1-colouring of K_{r-1}, for $k > 1$ take two disjoint copies of the construction for $k - 1$ and colour every edge between the two copies with a new colour. This construction led Bondy and Erdős [2] to make the following conjecture.

Conjecture 3. (Bondy, Erdős) *Equality holds in (1) for all odd integers $r > 3$.*

In this paper we disprove Conjecture 3 by using the result of Theorem 2 to construct colourings that give new lower bounds for $R_k(C_r)$ whenever r is an odd integer and k is sufficiently large.

Theorem 4. *For all odd integers r there exists a constant $\epsilon = \epsilon(r) > 0$ such that, for all k sufficiently large, $R_k(C_r) > (r-1)(2+\epsilon)^{k-1}$.*

The proof of Theorem 4 can be found in Section 3. We remark that Theorem 4 can not be used to say anything about the behaviour of $R_k(C_r)$ when k is fixed and r is increasing. Bondy and Erdős [2] showed that their conjecture holds for all r when $k = 2$. For $k = 3$, Łuczak [9] employed the regularity method to prove that Bondy and Erdős's