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On the structure of oriented graphs and digraphs with forbidden tournaments or cycles ☆

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Motivated by his work on the classification of countable homogeneous oriented graphs, Cherlin asked about the typical structure of oriented graphs (i) without a transitive triangle, or (ii) without an oriented triangle. We give an answer to these questions (which is not quite the predicted one). Our approach is based on the recent 'hypergraph containers' method, developed independently by Saxton and Thomason as well as by Balogh, Morris and Samotij. Moreover, our results generalise to forbidden transitive tournaments and forbidden oriented cycles of any order, and also apply to digraphs. Along the way we prove several stability results for extremal digraph problems, which we believe are of independent interest.

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1. Introduction

1.1. H-free graphs

Given a fixed graph H, a graph is called *H-free* if it does not contain H as a (not necessarily induced) subgraph. In 1976 Erdős, Kleitman and Rothschild [17] asymptotically determined the logarithm of the number of K_k-free graphs on n vertices, for every $k \geq 3$. This was strengthened by Kolaitis, Prömel and Rothschild [20], who showed that almost all K_k-free graphs are $(k-1)$-partite, for every $k \geq 3$ (the case $k = 3$ of this was already proved in [17]). This was one of the starting points for a vast body of work concerning the number and structure of H-free graphs on n vertices (see, e.g. [5–7,9,11,16,20,23,25]). The strongest of these results essentially state that for a large class of graphs $\mathcal{H}$, and any $H \in \mathcal{H}$, almost all H-free graphs have a similar structure to that of the extremal H-free graph. More recently, some related results have been proved for hypergraphs (see, e.g. [10,24]).

However, the corresponding questions for digraphs and oriented graphs are almost all wide open, and are the subject of this paper. Until now the only results of the above type for oriented graphs were proved by Balogh, Bollobás and Morris [3,4] who classified the possible 'growth speeds' of oriented graphs with a given property. Moreover Robinson [26,27], and independently Stanley [30], counted the number of acyclic digraphs. A related problem was considered by Alon and Yuster [2], who determined $\max_G D(G,T)$ over all n-vertex graphs G for sufficiently large n, where T is a fixed tournament and $D(G,T)$ denotes the number of T-free orientations of G (note that $\sum_G D(G,T)$ is the number of T-free oriented graphs on n vertices).

1.2. Oriented graphs and digraphs with forbidden tournaments or cycles

A *digraph* is a pair (V,E) where V is a set of vertices and E is a set of ordered pairs of distinct vertices in V (note that this means that in a digraph we do not allow loops or multiple edges in the same direction). An *oriented graph* is a digraph with at most one edge between two vertices, so may be considered as an orientation of a simple undirected graph. A *tournament* is an orientation of a complete graph. We denote a transitive tournament on k vertices by T_k, and a directed cycle on k vertices by C_k. We only consider labelled graphs and digraphs.

Given a class of graphs $\mathcal{A}$, we let $\mathcal{A}_n$ denote the set of all graphs in $\mathcal{A}$ that have precisely n vertices, and we say that *almost all graphs in $\mathcal{A}$ have property $\mathcal{B}$* if

$$\lim_{n\to\infty} \frac{|\{G \in \mathcal{A}_n : G \text{ has property } \mathcal{B}\}|}{|\mathcal{A}_n|} = 1.$$

Clearly any transitive tournament is C_k-free for any k, and any bipartite digraph is T_3-free. In 1998 Cherlin [13] gave a classification of countable homogeneous oriented