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Dual trees must share their ends

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ABSTRACT

We extend to infinite graphs the matroidal characterization of finite graph duality, that two graphs are dual iff they have complementary spanning trees in some common edge set. The naive infinite analogue of this fails.

The key in an infinite setting is that dual trees must share between them not only the edges of their host graphs but also their ends: the statement that a set of edges is acyclic and connects all the vertices in one of the graphs iff the remaining edges do the same in its dual will hold only once each of the two graphs' common ends has been assigned to one graph but not the other, and 'cycle' and 'connected' are interpreted topologically in the space containing the respective edges and precisely the ends thus assigned.

This property characterizes graph duality: if, conversely, the spanning trees of two infinite graphs are complementary in this end-sharing way, the graphs form a dual pair.

1. Introduction

It is well known (and easy to show) that two finite graphs are dual, in the usual sense that the circuits of one are the bonds of the other [8], if and only if they can be drawn with a common abstract set of edges so that the edge sets of the spanning trees of one are the complements of the edge sets of the spanning trees of the other:

URL: http://www.math.uni-hamburg.de/home/diestel/ (R. Diestel).

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Theorem 1. *Let $G = (V, E)$ and $G^* = (V^*, E)$ be connected finite graphs with the same abstract edge set. Then the following statements are equivalent:*

(i) *G and G^* are duals of each other.*
(ii) *Given any set $F \subseteq E$, the graph (V, F) is a tree if and only if $(V^*, F^\complement)$ is a tree.*

For infinite dual graphs G and G^* (see [4]), Theorem 1 (ii) will usually fail: when (V, F) is a spanning tree of G, the subgraph $(V^*, F^\complement)$ of G^* will be acyclic but may be disconnected. For example, consider as G the infinite $\mathbb{Z} \times \mathbb{Z}$ grid, and let F be the edge set of any spanning tree containing a two-way infinite path, a *double ray* R. Then the edges of R will form a cut in G^*, so $(V^*, F^\complement)$ will be disconnected.

Although the graphs $(V^*, F^\complement)$ in this example will always be disconnected, they become arc-connected (but remain *acirclic*) when we consider them as closed subspaces of the topological space obtained from G^* by adding its end. Such subspaces are called *topological spanning trees*; they provide the 'correct' analogues in infinite graphs of spanning trees in finite graphs for numerous problems, and have been studied extensively [9,10]. For $G = \mathbb{Z} \times \mathbb{Z}$, then, the complements of the edge sets of ordinary spanning trees of G form topological spanning trees in G^*, and vice versa (as $\mathbb{Z} \times \mathbb{Z}$ is self-dual).

It was shown recently in the context of infinite matroids [5] that this curious phenomenon is not specific to this example but occurs for all dual pairs of graphs: neither ordinary nor topological spanning trees permit, by themselves, an extension of Theorem 1 to infinite graphs, but as soon as one notion is used for G and the other for G^*, the theorem does extend. The purpose of this paper is to explain this seemingly odd phenomenon by a more general duality for graphs with ends, in which it appears as merely a pair of extreme cases.

It was shown in [6] that 2-connected dual graphs do not only have the 'same' edges but also the 'same' ends: there is a bijection between their ends that commutes with the bijection between their edges so as to preserve convergence of edges to ends.[1] Now if G and G^* are dual 2-connected graphs with edge sets E and end sets Ω, our result is that if we specify *any* subset Ψ of Ω and consider topological spanning trees of G in the space obtained from G by adding only the ends in Ψ, then Theorem 1 (ii) will hold if the subgraphs $(V^*, F^\complement)$ of G^* are furnished with precisely the ends in $\Omega \setminus \Psi$. (Our earlier example is the special case of this result with either $\Psi = \emptyset$ or $\Psi = \Omega$.) And conversely, if the spanning trees of two graphs G and G^* with common edge and end sets complement each other in this way for some—equivalently, for every—subset Ψ of their ends then G and G^* form a dual pair.

Here, then, is the formal statement of our theorem. A graph G is *finitely separable* if any two vertices can be separated by finitely many edges; as noted by Thomassen [13,14], this slight weakening of local finiteness is necessary for any kind of graph duality to be possible. The Ψ-*trees* in G, for subsets Ψ of its ends, will be defined in Section 2.

[1] See the end of Section 2 for a more formal definition.

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