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5-choosability of graphs with crossings far apart ☆

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ABSTRACT

We give a new proof of the fact that every planar graph is 5-choosable, and use it to show that every graph drawn in the plane so that the distance between every pair of crossings is at least 15 is 5-choosable. At the same time we may allow some vertices to have lists of size four only, as long as they are far apart and far from the crossings.

Thomassen [5] showed that every planar graph is 5-choosable. We prove a strengthening of this result, allowing some crossings. Suppose that a graph G is drawn in the plane with some crossings, and for $i \in \{1, 2\}$, let $u_i v_i$ and $x_i y_i$ be two edges of G crossing each other. The *distance* between the crossing of $u_1 v_1$ with $x_1 y_1$ and the crossing of $u_2 v_2$ with

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x_2y_2 is the length of the shortest path in G with one end in the set $\{u_1, v_1, x_1, y_1\}$ and the other end in the set $\{u_2, v_2, x_2, y_2\}$. Our main result is the following.

Theorem 1. *Every graph drawn in the plane so that the distance between every pair of crossings is at least* 15 *is* 5*-choosable.*

Let us recall that a *list assignment* L for G is a function that assigns to each vertex of G a set $L(v)$, called the *list of admissible colors* for v. An *L-coloring* is a choice of a color $\varphi(v) \in L(v)$ for each $v \in V(G)$ such that no two adjacent vertices receive the same color. The graph is *k-choosable* if it admits an L-coloring for every list assignment L with $|L(v)| \geq k$ for every $v \in V(G)$.

The main idea of Thomassen's beautiful proof of 5-choosability of planar graphs is to establish the following more general statement.

Theorem 2 *(Thomassen [5]). Let G be a plane graph with the outer face F, xy an edge of F, and L a list assignment such that $|L(v)| \geq 5$ for $v \in V(G) \setminus V(F)$, $|L(v)| \geq 3$ for $v \in V(F) \setminus \{x, y\}$, $|L(x)| = |L(y)| = 1$ and $L(x) \neq L(y)$. Then G is L-colorable.*

Let us note that the lists of x and y of size 1 give a precoloring of a path of length 1 in the outer face of G. Naturally, one might try to prove Theorem 1 by showing a variant of Theorem 2 allowing distant crossings. Unfortunately, almost any attempt to alter the statement of Theorem 2 (e.g., by allowing more than two vertices to be precolored, allowing lists of size 2 subject to some constraints, allowing some crossings in the drawing, etc.) fails with infinitely many counterexamples. To overcome this obstacle, we give a new proof of 5-choosability of planar graphs, see Theorem 6 in Section 1, which turns out to be more robust with respect to some strengthenings of the planar 5-choosability theorem. The proof of Theorem 6 is inspired by Thomassen's proof [6] of 3-choosability of planar graphs of girth 5.

In the course of the proof of Theorem 1, it is convenient to allow in addition to crossings also some other irregularities, such as vertices with fewer than 5 available colors. Hence, we actually obtain the following stronger statement (the distance between a vertex z and a crossing of edges uv and xy is the length of the shortest path from z to $\{u, v, x, y\}$).

Theorem 3. *Let G be a graph drawn in the plane with some crossings and let $N \subseteq V(G)$ be a set of vertices such that the distance between any pair of crossed edges is at least* 15*, the distance between any crossing and a vertex in N is at least* 13*, and the distance between any two vertices in N is at least* 11*. Then G is L-colorable for any list assignment L such that $|L(v)| = 4$ for $v \in N$ and $|L(v)| \geq 5$ for $v \in V(G) \setminus N$.*

Some distance condition on the crossings in Theorem 1 is necessary, even if we allow only three crossings, as shown by K_6. On the other hand, it was proved in [3] and independently also in [2] that the distance requirement is not needed, if we have at most two crossings. The inductive proof of Theorem 3 involves a stronger inductive hypothesis

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