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# Coloring graphs without fan vertex-minors and graphs without cycle pivot-minors

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ABSTRACT

A fan $F_k$ is a graph that consists of an induced path on $k$ vertices and an additional vertex that is adjacent to all vertices of the path. We prove that for all positive integers $q$ and $k$, every graph with sufficiently large chromatic number contains either a clique of size $q$ or a vertex-minor isomorphic to $F_k$. We also prove that for all positive integers $q$ and $k \geq 3$, every graph with sufficiently large chromatic number contains either a clique of size $q$ or a pivot-minor isomorphic to a cycle of length $k$.



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## 1. Introduction

All graphs in this paper are simple, which means that they have no loops and no parallel edges. Given a graph, a *clique* is a set of pairwise adjacent vertices and an *independent set* is a set of pairwise non-adjacent vertices. For a graph $G$, let $\chi(G)$ denote the *chromatic number* of $G$ and let $\omega(G)$ denote the maximum size of a clique of $G$. Since two vertices in a clique cannot receive the same color in a proper coloring, the clique number is a trivial lower bound for the chromatic number. If $\chi(H) = \omega(H)$ for every induced subgraph $H$ of a graph $G$, then we say $G$ is *perfect*. Gyárfás [20] introduced the notion of a $\chi$-bounded class as a generalization of perfect graphs. A class $\mathcal{C}$ of graphs is *$\chi$-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for all graphs $G \in \mathcal{C}$, and all induced subgraphs $H$ of $G$, $\chi(H) \leq f(\omega(H))$. Therefore the class of perfect graphs is $\chi$-bounded with the identity function.

*Vertex-minors* and *pivot-minors* are graph containment relations introduced by Bouchet [4–7] while conducting research on circle graphs (intersection graphs of chords on a cycle) and 4-regular Eulerian digraphs, and these graph operations have been used for developing theory on rank-width [2,21,27–30]. Given a graph $G$ and a vertex $v$ of $G$, let $G * v$ denote the graph obtained from $G$ by applying local complementation at $v$; *local complementation* at $v$ is an operation to replace the subgraph induced on the neighborhood of $v$ with its complement. The graph obtained from $G$ by *pivoting* an edge $uv$ of $G$ is defined as $G \wedge uv := G * u * v * u$. We provide an example of pivoting in Section 2. A graph $H$ is a *vertex-minor* of $G$ if $H$ can be obtained from $G$ by applying a sequence of local complementations and vertex deletions, and a graph $H$ is a *pivot-minor* of $G$ if $H$ can be obtained from $G$ by a sequence of pivoting edges and deleting vertices. We note that for every graph $H$, any subdivision of $H$ contains $H$ as a vertex-minor, since we can simulate the reverse operation of subdividing an edge by applying local complementation at a vertex of degree 2 and then removing the vertex.

Chudnovsky, Robertson, Seymour, and Thomas [10] proved the strong perfect graph theorem, which states that a graph $G$ is perfect if and only if neither $G$ nor its complement contains an induced odd cycle of length at least 5. This shows that there is a deep connection between the chromatic number and the structure of the graph. Gyárfás [20] proved that for each integer $k$, the class of graphs with no induced path of length $k$ is $\chi$-bounded. Gyárfás also made the following three conjectures for $\chi$-boundedness in terms of forbidden induced subgraphs. Note that (iii) implies both (i) and (ii).

**Conjecture 1.1** *(Gyárfás [20]). The following classes are $\chi$-bounded:*

(i) *The class of graphs with no induced odd cycle of length at least* 5.
(ii) *The class of graphs with no induced cycle of length at least $k$ for a fixed $k$.*
(iii) *The class of graphs with no induced odd cycle of length at least $k$ for a fixed $k$.*