ARTICLE IN PRESS YJCTB:3050

Journal of Combinatorial Theory, Series B ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Notes

On edges not in monochromatic copies of a fixed bipartite graph

Jie Ma [1]

School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China

ARTICLE INFO

Article history:
Received 25 May 2016
Available online xxxx

Keywords:
Turán number
Bipartite graph
Monochromatic copies of H
2-Edge-coloring of K_n
Weak vertex

ABSTRACT

Let H be a fixed graph. Let $f(n, H)$ be the maximum number of edges not contained in any monochromatic copy of H in a 2-edge-coloring of the complete graph K_n, and $ex(n, H)$ be the *Turán number* of H. An easy lower bound shows $f(n, H) \geq ex(n, H)$ for any H and n. In [9], Keevash and Sudakov proved that if H is an edge-color-critical graph or C_4, then $f(n, H) = ex(n, H)$ holds for large n, and they asked if this equality holds for any graph H when n is sufficiently large. In this paper, we provide an affirmative answer to this problem for an abundant infinite family of bipartite graphs H, including all even cycles and complete bipartite graphs $K_{s,t}$ for $t > s^2-3s+3$ or $(s, t) \in \{(3, 3), (4, 7)\}$. In addition, our proof shows that for all such H, the 2-edge-coloring c of K_n achieves the maximum number $f(n, H)$ if and only if one of the color classes in c induces an extremal graph for $ex(n, H)$. We also obtain a multi-coloring generalization for bipartite graphs. Some related problems are discussed in the final section.

E-mail address: jiema@ustc.edu.cn.

[1] Partially supported by NSFC projects 11501539 and 11622110.

http://dx.doi.org/10.1016/j.jctb.2016.11.006
0095-8956/

Please cite this article in press as: J. Ma, On edges not in monochromatic copies of a fixed bipartite graph, J. Combin. Theory Ser. B (2016), http://dx.doi.org/10.1016/j.jctb.2016.11.006

1. Introduction

Given a graph H, let $f(n, H)$ be the maximum number of edges not contained in any monochromatic copy of H in a 2-edge-coloring of the complete graph K_n, and let $ex(n, H)$ be the *Turán number* of H, i.e., the maximum number of edges in an n-vertex H-free graph. The problem of determining $f(n, H)$ was motivated by counting the number of monochromatic cliques, and we refer interested readers to [9] for a thoughtful discussion on the background and related topics. (For results on monochromatic cliques, see [7,14,4,5,8,15,3].)

If one considers the 2-edge-coloring of K_n in which one of the colors induces the largest H-free graph, then it is easy to see that for any H and n, we have

$$f(n, H) \geq ex(n, H). \tag{1}$$

Erdős, Rousseau and Schelp (see [4]) showed that $f(n, K_3) = ex(n, K_3)$ for sufficiently large n, and this also can be derived from a result of Pyber in [12] for $n \geq 2^{1500}$. The generalization of this result was suggested by Erdős in [4]. Keevash and Sudakov [9] studied general graphs and asked that if, for large n, the above lower bound (1) is tight.

Problem 1.1. ([9]) Let H be a fixed graph. Is it true that for n sufficiently large, $f(n, H) = ex(n, H)$?

A graph H is *edge-color-critical*, if there exists an edge $e \in E(H)$ such that $\chi(H-e) < \chi(H)$. The authors of [9] confirmed it for H being any edge-color-critical graph or a C_4, and in fact, quite amazingly, they were able to determine the value of $f(n, H)$ for every n when H is a K_3 or C_4. We quote from their remark [9] that "for bipartite graphs the situation is less clear, as even the asymptotics of the Turán numbers are known only in a few cases".

In this paper, we provide an affirmative answer to Problem 1.1 for an abundant infinite family of bipartite graphs. A vertex w in a bipartite graph H is called *weak*, if

$$ex(n, H - w) = o(ex(n, H)).$$

The notation of weak vertices is explicitly defined in the literature and has been well studied (see [13]). We call a bipartite graph H *reducible*, if it contains a weak vertex w such that $H - w$ is connected. For instance all even cycles are reducible. Our main theorem is as follows.

Theorem 1.2. *Let H be a reducible bipartite graph. Then for sufficiently large n, $f(n, H) = ex(n, H)$. Moreover, a 2-edge-coloring of K_n achieves the maximum number $f(n, H)$ if and only if one of the color classes induces an extremal graph for $ex(n, H)$.*

We point out that the "moreover" part is new for C_4, while its analog is not true for edge-color-critical graphs as noticed in [9].