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A non-trivial upper bound on the threshold bias of the Oriented-cycle game

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ABSTRACT

In the Oriented-cycle game, introduced by Bollobás and Szabó [7], two players, called OMaker and OBreaker, alternately direct edges of K_n. OMaker directs exactly one previously undirected edge, whereas OBreaker is allowed to direct between one and b previously undirected edges. OMaker wins if the final tournament contains a directed cycle, otherwise OBreaker wins. Bollobás and Szabó [7] conjectured that for a bias as large as $n-3$ OMaker has a winning strategy if OBreaker must take exactly b edges in each round. It was shown recently by Ben-Eliezer, Krivelevich and Sudakov [6], that OMaker has a winning strategy for this game whenever $b < n/2 - 1$. In this paper, we show that OBreaker has a winning strategy whenever $b > 5n/6 + 1$. Moreover, in case OBreaker is required to direct exactly b edges in each move, we show that OBreaker wins for $b \geqslant 19n/20$, provided that n is large enough. This refutes the conjecture by Bollobás and Szabó.

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1. Introduction

We consider biased orientation games, as discussed by Ben-Eliezer, Krivelevich and Sudakov in [6]. In orientation games, the board consists of the edges of the complete graph K_n. In the $(a : b)$ orientation game, the two players called OMaker and OBreaker, alternately direct previously undirected edges. OMaker starts, and in each round, OMaker directs between one and a edges, and then OBreaker directs between one and b edges. At the end of the game, the final graph is a tournament on n vertices. OMaker wins the game if this tournament has some predefined property $\mathcal{P}$. Otherwise, OBreaker wins.

Orientation games can be seen as a relative of $(a : b)$ Maker–Breaker games, played on the complete graph K_n. The game is played by two players, Maker and Breaker, who alternately claim a and b edges, respectively. Maker wins if the subgraph consisting of her edges satisfies some given monotone-increasing property $\mathcal{P}$. Otherwise, Breaker wins. Maker–Breaker games have been widely studied (cf. [1–3,5,8,10,11,14]), and it is quite natural to translate typical questions about Maker–Breaker games to orientation games.

For instance, Beck [4] studied the so-called *Clique game*, proving that in the $(1 : 1)$ Maker–Breaker game, the largest clique that Maker is able to build is of size $(2 - o(1)) \log_2(n)$. Motivated by this result, an orientation game version of the Clique game is considered in [9]: Given a tournament T_k on k vertices, it is proven there that for $k \leqslant (2 - o(1)) \log_2(n)$ OMaker can ensure that T_k appears in the final tournament, while for $k \geqslant (4 + o(1)) \log_2(n)$ OBreaker always can prevent a copy of T_k.

In this work we only consider orientation games with $a = 1$. We refer to the $(1 : 1)$ orientation game as the *unbiased* orientation game, and the $(1 : b)$ orientation game as the *b-biased orientation game* when $b > 1$. Increasing b can only help OBreaker (since OBreaker can choose to direct fewer than b edges per round) so the game is *bias monotone*. Therefore, any such game (besides degenerate games where $\mathcal{P}$ is a property that is satisfied by every or by no tournament on n vertices) has a *threshold* $t(n, \mathcal{P})$ such that OMaker wins the b-biased game when $b \leqslant t(n, \mathcal{P})$ and OBreaker wins the game when $b > t(n, \mathcal{P})$.

In a variant, OBreaker is required to direct exactly b edges in each round. We refer to this variant as the *strict* b-biased orientation game, where *the strict rules apply*. Accordingly, we say the *monotone rules apply* in the game we defined above – when OBreaker is free to direct between one and b edges. Playing the exact bias in every round may be disadvantageous for OBreaker, so the existence of a threshold as for the monotone rules is unclear in general. We therefore define $t^+(n, \mathcal{P})$ to be the largest value b such that OMaker has a strategy to win the strict b-biased orientation game, and $t^-(n, \mathcal{P})$ to be the largest integer such that for *every* $b \leqslant t^-(n, \mathcal{P})$, OMaker has a strategy to win the strict b-biased orientation game. (The definition of these two threshold functions is motivated by the study of Avoider–Enforcer games, cf. [13,12].) Trivially, $t(n, \mathcal{P}) \leqslant t^-(n, \mathcal{P}) \leqslant t^+(n, \mathcal{P})$ holds.