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# A new proof of Seymour's 6-flow theorem

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ABSTRACT

Tutte's famous 5-flow conjecture asserts that every bridgeless graph has a nowhere-zero 5-flow. Seymour proved that every such graph has a nowhere-zero 6-flow. Here we give (two versions of) a new proof of Seymour's Theorem. Both are roughly equal to Seymour's in terms of complexity, but they offer an alternative perspective which we hope will be of value.


## 1. Introduction

By default we permit graphs to have parallel edges, and we forbid loops. Let $G$ be a graph which is equipped with an orientation of its edges and let $\Gamma$ be an additively written abelian group. We define a function $\varphi : E(G) \to \Gamma$ to be a $\Gamma$-*flow* if the following

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condition, called *Kirchhoff's law*, is satisfied at every vertex $v$. Here $\delta^+(v)$ ($\delta^-(v)$) denotes the set of edges directed away from (toward) $v$.

$$\sum_{e \in \delta^+(v)} \varphi(e) = \sum_{e \in \delta^-(v)} \varphi(e)$$

We say that $\varphi$ is *nowhere-zero* if $0 \notin \varphi(E(G))$ and for a positive integer $k$ we say that $\varphi$ is a *$k$-flow* if $\Gamma = \mathbb{Z}$ and $|\varphi(e)| < k$ holds for every edge $e$. Note that if $\varphi$ is a $\Gamma$-flow, we may reverse the orientation of an edge $e$ and replace $\varphi(e)$ by its additive inverse to obtain a new flow. This operation preserves the properties of being nowhere-zero and $k$-flow, so the question of whether $G$ has a nowhere-zero $\Gamma$-flow ($k$-flow) depends only on the underlying graph and not the particular orientation. Accordingly, we say that an undirected graph $G$ has a nowhere-zero $\Gamma$-flow ($k$-flow) if some (and thus every) orientation of $G$ admits such a function.

Tutte initiated the study of nowhere-zero flows by demonstrating that nowhere-zero $k$-flows are dual to $k$-colourings in planar graphs. He made three famous conjectures, known as the 5-flow, 4-flow, and 3-flow conjectures which extend various colouring theorems about planar graphs to arbitrary graphs. These conjectures have been the driving motivation for the subject, and all three remain open despite considerable effort. For the purposes of this paper, the relevant conjecture is the 5-flow conjecture. The reader interested to learn more about the area may consult [1, Chapter 6] or [6].

**Conjecture 1** *(Tutte [5]). Every 2-edge-connected graph has a nowhere-zero 5-flow.*

The Petersen graph does not have a nowhere-zero 4-flow, so if true, this conjecture is best possible. The strongest partial result toward this conjecture is Seymour's 6-flow theorem.

**Theorem 2** *(Seymour [4]). Every 2-edge-connected graph has a nowhere-zero 6-flow.*

The purpose of this paper is to provide a new proof of this theorem. In fact we will give two (closely related) variants of this new proof. Seymour's original paper also contains two proofs of the 6-flow theorem which are similar in some ways, but have somewhat different character. Both of our proofs are roughly equal to Seymour's in terms of complexity, but they offer an alternative perspective which we hope will be of value. All of these proofs rely on the following key result of Tutte.

**Theorem 3** *(Tutte). For every graph $G$ and positive integer $k$ we have the following equivalence: $G$ has a nowhere-zero $k$-flow if and only if $G$ has a nowhere-zero $\mathbb{Z}_k$-flow.*

Though we shall not require it, let us note that Tutte proved more generally that for every abelian group $\Gamma$ of order $k$, a graph has a nowhere-zero $k$-flow if and only if it has a nowhere-zero $\Gamma$-flow. For our purposes, the key consequence of Theorem 3 (and