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Cayley numbers with arbitrarily many distinct prime factors

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ABSTRACT

A positive integer n is a Cayley number if every vertex-transitive graph of order n is a Cayley graph. In 1983, Dragan Marušič posed the problem of determining the Cayley numbers. In this paper we give an infinite set S of primes such that every finite product of distinct elements from S is a Cayley number. This answers a 1996 outstanding question of Brendan McKay and Cheryl Praeger, which they "believe to be the key unresolved question" on Cayley numbers. We also show that, for every finite product n of distinct elements from S, every transitive group of degree n contains a semiregular element.

1. Introduction

In this paper, all groups considered are finite and all graphs and digraphs are finite and have no multiple edges or arcs. (They may have loops and they may be disconnected.) A (di)graph Γ is **vertex-transitive** if the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ acts transitively on the vertices of Γ. Let R be a group and let S be a subset of R, the **Cayley digraph** on

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R with connection set S, denoted $\mathrm{Cay}(R, S)$, is the digraph with vertex-set R and with (g, h) being an arc if and only if $gh^{-1} \in S$. It is easy to see that $\mathrm{Cay}(R, S)$ is a graph if and only if S is inverse-closed, that is, $S = \{s^{-1} \mid s \in S\}$. It is also clear that the action of R on itself by right multiplication gives rise to a group of automorphisms of $\mathrm{Cay}(R, S)$ which is transitive on its vertex-set; thus $\mathrm{Cay}(R, S)$ is vertex-transitive. With a slight abuse of terminology, we say that a (di)graph Γ is a Cayley (di)graph, if Γ is isomorphic to some Cayley (di)graph: it is well-known and easy to prove [17, Lemma 4] that this happens if and only if $\mathrm{Aut}(\Gamma)$ contains a subgroup R with R transitive on the vertices of Γ and with the identity being the only element of R fixing some vertex of Γ.

A positive integer n is called a **Cayley number** if every vertex-transitive graph of order n is a Cayley graph. In 1983, Dragan Marušič [14] asked for a concrete determination of the positive integers n for which every vertex-transitive graph of order n is a Cayley graph, that is, Marušič posed the problem of determining Cayley numbers. Clearly, every prime number is a Cayley number, and 10 is not a Cayley number because the Petersen graph is not a Cayley graph.

The question of Marušič has generated a fair amount of interest, with most results giving explicit numbers that are not Cayley. In fact, it seems to be easier to provide integers that are not Cayley numbers: to show that n is not a Cayley number it suffices to exhibit one single vertex-transitive graph of order n that is not a Cayley graph, but to show that n is a Cayley number one needs to prove that each vertex-transitive graph of order n is a Cayley graph.

It is useful to observe that the integers which are not Cayley numbers, called **non-Cayley numbers**, are closed under multiplication, that is, if n is a non-Cayley number and m is a positive integer, then nm is also a non-Cayley number. This is easy to see as, if Γ is a graph of order n that is not a Cayley graph, then for every positive integer m the disjoint union of m copies of Γ yields a graph of order nm that is also not a Cayley graph. The reader is referred to [11,15,16,18] for many examples of non-Cayley numbers and to [3,5,9,10,14] for work towards establishing that certain integers are Cayley numbers.

In this paper, we answer a question of Brendan McKay and Cheryl Praeger [16, Question] that they "believe to be the key unresolved question" concerning the structure of the set of Cayley numbers; namely, there exist Cayley numbers that are product of arbitrarily many distinct primes.

Theorem 1.1. *There exists an infinite set S of primes such that for every finite product n of distinct elements from S, every vertex-transitive digraph of order n is a Cayley digraph. Consequently, n is a Cayley number and there exist Cayley numbers that are a product of arbitrarily many distinct primes.*

The proof of Theorem 1.1 follows from Theorem 1.2, which we believe to be of independent interest. (A transitive permutation group G is **quasiprimitive** if every non-identity normal subgroup of G is transitive.)