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A simple arithmetic criterion for graphs being determined by their generalized spectra ☆

Wei Wang

School of Mathematics and Statistics, Xi'an Jiaotong University, No. 28 Xianning West Rd., Xi'an, Shaanxi, 710049, PR China

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ABSTRACT

A graph G is said to be determined by its generalized spectrum (DGS for short) if, whenever H is a graph such that H and G are cospectral with cospectral complements, then H must be isomorphic to G.

It turns out that whether a graph G is DGS is closely related to the arithmetic properties of its walk-matrix. More precisely, let A be the adjacency matrix of a graph G on n vertices, and let $W = [e, Ae, A^2e, \cdots, A^{n-1}e]$ (e is the all-ones vector) be its *walk-matrix*. In Wang (2013) [16], the author defined a large family of graphs

$$\mathcal{F}_n = \{G \mid \frac{\det(W)}{2^{\lfloor \frac{n}{2} \rfloor}} \text{ is an odd square-free integer}\}$$

(which may have positive density among all graphs, as suggested by some numerical experiments) and conjectured every graph in $\mathcal{F}_n$ is DGS.

In this paper, we show that the conjecture is actually true, thereby giving a simple arithmetic condition for determining whether a graph is DGS.

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E-mail address: wang_weiw@163.com.

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1. Introduction

The spectra of graphs encode a lot of combinatorial information about the given graphs, and thus have long been a useful tool in dealing with various problems in graph theory, even if they have nothing to do with graph spectra in the appearance.

A fundamental question in the theory of graph spectra is: "What kinds of graphs are determined by the spectrum (DS for short)?" The problem dates back to more than 50 years and originates from chemistry and has received a lot of attention from researchers in recent years.

In 1956, Günthard and Primas [8] raised the question in a paper that relates the theory of graph spectra to Hückel's theory from chemistry. At that time it was believed that every graph is DS until one year later Collatz and Sinogowitz [2] presented a pair of cospectral trees. Since then, various constructions of cospectral graphs (i.e., graphs having the same spectrum) have been studied extensively and many results are obtained in literature. For example, Godsil and McKay [7] invented a powerful method called GM-switching, which can produce lots of pairs of cospectral graphs (with cospectral complements). Another important result was given by Schwenk [12], stating that almost all trees are not DS.

It is worthwhile to mention that the above problem is also closely related to a famous problem of Kac [9]: "Can one hear the shape of a drum?" Fisher [5] modeled the shape of the drum by a graph. Then the sound of that drum is characterized by the eigenvalues of the graph. Thus Kac's question is essentially the same as ours.

However, it turns out that proving graphs to be DS is more difficult than constructing cospectral graphs. Up to now, all the known DS graphs have very special properties, and the techniques (e.g., the eigenvalue interlacing technique) involved in proving them to be DS depend heavily on some special properties of the spectra of these graphs, and cannot be applied to general graphs. For the background and some known results about this problem, we refer the reader to [13,14] and the references therein.

The above problem clearly depends on the spectrum concerned. In [17,18], Wang and Xu gave a method for determining whether a graph G is determined by its generalized spectrum (DGS for short, see Section 2 for details), which works for a large family of general graphs. We briefly describe their method below.

Let G and H be two graphs that are cospectral with cospectral complements. Then there exists an orthogonal matrix Q with $Qe = e$ (e is the all-ones vector) such that $Q^{\mathrm{T}}A(G)Q = A(H)$, where $A(G)$ and $A(H)$ are the adjacency matrices of G and H, respectively. Moreover, the matrix Q can be chosen to be a rational matrix (under mild restrictions). Thus, if we can show that every rational orthogonal matrix Q with $Qe = e$ such that $Q^{\mathrm{T}}A(G)Q$ is a $(0, 1)$-matrix with zero diagonal must be a permutation matrix, then G is clearly DGS. This seems, at first glance, as difficult as the original problem. However, the authors managed to find some algorithmic methods to achieve this goal, by using some arithmetic properties of the walk-matrix associated with the given graph.

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