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Turán problems and shadows II: Trees

Alexandr Kostochka [a,b,1], Dhruv Mubayi [c,2], Jacques Verstraëte [d,3]

[a] *University of Illinois at Urbana–Champaign, Urbana, IL 61801, United States*
[b] *Sobolev Institute of Mathematics, Novosibirsk 630090, Russia*
[c] *Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, United States*
[d] *Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0112, USA*

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A B S T R A C T

The expansion G^+ of a graph G is the 3-uniform hypergraph obtained from G by enlarging each edge of G with a vertex disjoint from $V(G)$ such that distinct edges are enlarged by distinct vertices. Let $\mathrm{ex}_r(n, F)$ denote the maximum number of edges in an r-uniform hypergraph with n vertices not containing any copy of F. The authors [10] recently determined $\mathrm{ex}_3(n, G^+)$ when G is a path or cycle, thus settling conjectures of Füredi–Jiang [8] (for cycles) and Füredi–Jiang–Seiver [9] (for paths).

Here we continue this project by determining the asymptotics for $\mathrm{ex}_3(n, G^+)$ when G is any fixed forest. This settles a conjecture of Füredi [7]. Using our methods, we also show that for any graph G, either $\mathrm{ex}_3(n, G^+) \le \left(\frac{1}{2} + o(1)\right) n^2$ or $\mathrm{ex}_3(n, G^+) \ge (1 + o(1))n^2$, thereby exhibiting a jump for the Turán number of expansions.

E-mail addresses: kostochk@math.uiuc.edu (A. Kostochka), mubayi@uic.edu (D. Mubayi), jverstra@math.ucsd.edu (J. Verstraëte).

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1. Introduction

An r-uniform hypergraph F, or simply *r-graph*, is a family of r-element subsets of a finite set. We associate an r-graph F with its edge set and call its vertex set $V(F)$. Given a set of r-graphs $\mathcal{F}$, let $\mathrm{ex}_r(n, \mathcal{F})$ denote the maximum number of edges in an r-graph on n vertices that does not contain any r-graph from $\mathcal{F}$. When $\mathcal{F} = \{F\}$ we write $\mathrm{ex}_r(n, F)$. We will omit the subscript r in this notation if it is obvious from context, and this paper deals exclusively with the case $r = 3$. Let G be a graph, and for each edge $e \in G$ let X_e be a set of $r - 2$ vertices so that $X_e \cap V(G) = \emptyset$ and $X_e \cap X_f = \emptyset$ when $e \neq f$. The r-uniform *expansion* G^+ of a graph G is the r-graph $G^+ = \{e \cup X_e : e \in G\}$.

Expansions of graphs include many important hypergraphs whose extremal functions have been investigated, for instance when G is a triangle and more generally a clique [7–13]. Even the simplest case of the expansion of a path with two edges is non-trivial, in this case we are asking for two hyperedges intersecting in exactly one point. Here the extremal function was determined by Frankl [5], settling a conjecture of Erdős and Sós. If a graph is not r-colorable then its r-uniform expansion G^+ is not r-partite, so $\mathrm{ex}_r(n, G^+) = \Omega(n^r)$. We focus on $\mathrm{ex}_r(n, G^+)$ when G is r-partite, where a well-known result of Erdős [1] yields $\mathrm{ex}(n, G^+) = O(n^{r-\epsilon_G})$ for some $\epsilon_G > 0$.

The authors [10] had previously determined $\mathrm{ex}_3(n, G^+)$ exactly (for large n) when G is a path or cycle of fixed length $k \geq 3$, thereby answering questions of Füredi–Jiang–Seiver [9] and Füredi–Jiang [8].

1.1. Results

A set of vertices in a hypergraph F containing exactly one vertex from every edge of F is called a *crosscut*, following Frankl and Füredi [6]. Let $\sigma(F)$ be the minimum size of a crosscut of F if it exists, i.e.,

$$\sigma(F) := \min\{|X| : X \subset V(F), \forall e \in F, |e \cap X| = 1\}$$

if such an X exists. We observe that crosscuts always exist for expansions.

Since the r-graph on n vertices consisting of all edges containing exactly one vertex from a fixed subset of size $\sigma(F) - 1$ does not contain F, we have

$$\mathrm{ex}_r(n, F) \geq (\sigma(F) - 1)\binom{n - \sigma(F) + 1}{r - 1} \sim (\sigma(F) - 1 + o(1))\binom{n}{r - 1}. \tag{1}$$

An intriguing open question is when asymptotic equality holds above and this is one of our motivations for this project. Indeed, it appears that the parameter $\sigma(F)$ often plays a crucial role in determining the extremal function for F. The value of $\mathrm{ex}_3(n, G^+)$ was determined precisely by the authors [10] when G is a path or cycle. Füredi [7] determined the asymptotics when G is a forest and $r \geq 4$, by showing that $\mathrm{ex}_r(n, G^+) = (\sigma(G^+) - 1 + o(1))\binom{n}{r-1}$. Füredi's proof involved extensive use of the delta system method