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Notes

When does the list-coloring function of a graph equal its chromatic polynomial [☆]

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ABSTRACT

Let G be a connected graph with n vertices and m edges. Using Whitney's broken cycle theorem, we prove that if $k > \frac{m-1}{\ln(1+\sqrt{2})} \approx 1.135(m-1)$ then for every k-list assignment L of G, the number of L-colorings of G is at least that of ordinary k-colorings of G. This improves previous results of Donner (1992) and Thomassen (2009), who proved the result for k sufficiently large and $k > n^{10}$, respectively.

1. Introduction

For a positive integer k, a *k-list assignment* of a graph $G = (V(G), E(G))$ is a mapping L which assigns to each vertex v a set $L(v)$ of k permissible colors. Given a k-list assignment L, an *L-list-coloring*, or *L-coloring* for short, is a mapping $c\colon V(G) \to \cup_{v\in V(G)} L(v)$ such that $c(v) \in L(v)$ for each vertex v, and $c(u) \neq c(v)$ for any two adjacent vertices u and v. The notion of list coloring was introduced by Vizing [6] as well as by Erdős, Rubin and Taylor [3].

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For a k-list assignment L, we use $P(G, L)$ to denote the number of L-colorings of G and, moreover, we use $P_l(G, k)$ to denote the minimum value of $P(G, L)$ over all k-list assignments L of G. We note that, if $L(v) = \{1, 2, \ldots, k\}$ for all vertices $v \in V(G)$, then an L-coloring is exactly an ordinary k-coloring [5] and therefore, $P(G, L)$ agrees with the classic *chromatic polynomial* $P(G, k)$ introduced by Birkhoff [1] in 1912. In this sense, $P_l(G, k)$ is an analogue of the chromatic polynomial. However, it was shown that $P_l(G, k)$ is in general not a polynomial [2], answering the problem of Kostochka and Sidorenko [4]. Following [5], we call $P_l(G, k)$ the *list-coloring function* of G. This leads to an interesting question: 'When does the list-coloring function $P_l(G, x)$ equal the chromatic polynomial $P(G, x)$ evaluated at k'. In [4] Kostochka and Sidorenko observed that if G is a chordal graph then $P_l(G, k) = P(G, k)$ for any positive integer k. For a general graph G, Donner [2] and Thomassen [5] proved that $P_l(G, k) = P(G, k)$ when k is sufficiently large. More specifically, Thomassen proved that $P_l(G, k) = P(G, k)$ provided $k > |V(G)|^{10}$.

In this note, we use Whitney's broken cycle theorem to prove the following result.

Theorem 1. *For any connected graph G with m edges, if*

$$k > \frac{m-1}{\ln(1+\sqrt{2})} \approx 1.135(m-1) \tag{1}$$

then $P_l(G, k) = P(G, k)$.

2. Proof of Theorem 1

Let G be a connected graph G with n vertices and m edges. Note that if $m \leq 1$ then G is K_1 or K_2 and Theorem 1 trivially holds. In what follows we assume $m \geq 2$ and, for the convenience of discussion, we label these m edges by $1, 2, \ldots, m$.

A *broken cycle* of G is a set of edges obtained from the edge set of a cycle of G by removing its maximum edge. Define a set system

$$\mathcal{B}(G) = \{S\colon\ S \subseteq E(G) \text{ and } S \text{ contains no broken cycle}\}. \tag{2}$$

Such a system is also called a broken circuit complex; see [8] for details. We note that any cycle contains at least one broken cycle. So for each $S \in \mathcal{B}(G)$, the spanning subgraph $(V(G), S)$ (the graph with vertex set $V(G)$ and edge set S) contains no cycles and hence $|S| \leq n-1$. We write

$$\mathcal{B}(G) = \mathcal{B}_0(G) \cup \mathcal{B}_1(G) \cup \cdots \cup \mathcal{B}_{n-1}(G), \tag{3}$$

where $\mathcal{B}_i(G) = \{S \in \mathcal{B}(G)\colon\ |S| = i\}$. Note that for any $S \in \mathcal{B}_i(G)$, the subgraph $(V(G), S)$ has exactly $n-i$ components, all of which are trees. Now Whitney's broken cycle theorem can be stated as follows.

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