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The number of colorings of planar graphs with no separating triangles

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ABSTRACT

A classical result of Birkhoff and Lewis implies that every planar graph with n vertices has at least $15 \cdot 2^{n-1}$ distinct 5-vertex-colorings. Equality holds for planar triangulations with $n - 4$ separating triangles. We show that, if a planar graph has no separating triangle, then it has at least $(2 + 10^{-12})^n$ distinct 5-vertex-colorings. A similar result holds for k-colorings for each fixed $k \geq 5$. Infinitely many planar graphs without separating triangles have less than 2.252^n distinct 5-vertex-colorings. As an auxiliary result we provide a complete description of the infinite 6-regular planar triangulations.

1. Introduction

If G is a graph and k is a nonnegative integer, then a k*-coloring* of G is a vertex-coloring where the colors are taken from the set $\{1, 2, \ldots, k\}$, and neighboring vertices get different colors. The *chromatic polynomial* $P(G, k)$ is the number of k-colorings of G. (In this

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definition, permuting the colors gives rise to a new coloring. Not all colors in $\{1, 2, \ldots, k\}$ are necessarily used.) The chromatic polynomial was introduced by Birkhoff [1] in 1912.

An old (still unsettled) conjecture by Birkhoff and Lewis [2] says that a planar graph has no chromatic root greater than or equal to 4. In fact, they proposed the following stronger conjecture.

Conjecture 1. *Let G be a planar graph on n vertices. Then, for each real number $k \geq 4$, $P(G, k) \geq k(k-1)(k-2)(k-3)^{n-3}$.*

If true, this would be best possible, as demonstrated by any planar triangulation obtained from K_4 by adding successively vertices of degree 3. Such a triangulation has $n-4$ separating triangles. And, when a graph G has a separating triangle, then it can be written as the union of two graphs G_1, G_2 having precisely that triangle in common. In this case the chromatic polynomial factorizes, namely

$$P(G, k) = P(G_1, k)P(G_2, k)/k(k-1)(k-2).$$

Therefore, planar graphs with no separating triangles are particularly interesting in this context. And one might hope that a method which improves Conjecture 1 for $k \geq 5$ might also be applicable for the interval from 4 to 5. In contrast to this, Royle [7] has shown that planar graphs have chromatic real root arbitrarily close to 4.

Birkhoff and Lewis verified Conjecture 1 for all real numbers ≥ 5. In particular, every planar graph with n vertices has at least $15 \cdot 2^{n-1}$ distinct 5-vertex-colorings. We show that, if it has no separating triangle, then it has at least $(2 + 10^{-12})^n$ distinct 5-vertex-colorings. There is a similar strengthening of Conjecture 1 for each natural number > 5, and hence for each real number > 5, as the chromatic polynomial increases for $k > 5$. Infinitely many planar graphs with no separating triangles have less than 2.252^n distinct 5-vertex-colorings.

The ideas in the proof are the following. We first investigate planar triangulations with no separating triangles and with many vertices of degree < 6. We show that such graphs have many 5-colorings. Then we turn to triangulations with few vertices of degree < 6. Euler's formula implies that such a graph has a vertex v such that all vertices of distance at most 100, say, from v induce a subgraph H where all vertices have degree precisely 6. We show that v can be chosen such that H can be extended to an infinite 6-regular planar triangulation. We characterize completely the infinite 6-regular triangulations. Thus we know the structure of H and we can apply that to improve Birkhoff and Lewis' lower bound on 5-colorings (in fact on k-colorings when $k \geq 5$) provided there are no separating triangles.

The terminology and notation are essentially the same as [3–5,8]. A *graph* has no loops or multiple edges. A *multigraph* may have multiple edges but no loops. If G is a multigraph, and S is a set of vertices, then the subgraph $G(S)$ induced by S has vertex set S and contains precisely those edges in G which join two vertices of S. If v is a vertex