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Minors and dimension ☆

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ABSTRACT

It has been known for 30 years that posets with bounded height and with cover graphs of bounded maximum degree have bounded dimension. Recently, Streib and Trotter proved that dimension is bounded for posets with bounded height and planar cover graphs, and Joret et al. proved that dimension is bounded for posets with bounded height and with cover graphs of bounded tree-width. In this paper, it is proved that posets of bounded height whose cover graphs exclude a fixed topological minor have bounded dimension. This generalizes all the aforementioned results and verifies a conjecture of Joret et al. The proof relies on the Robertson–Seymour and Grohe–Marx graph structure theorems.

1. Introduction

In this paper, we are concerned with finite partially ordered sets, which we simply call *posets*. The *dimension* of a poset P is the minimum number of linear orders that form a *realizer* of P, that is, their intersection gives rise to P. The notion of dimension was

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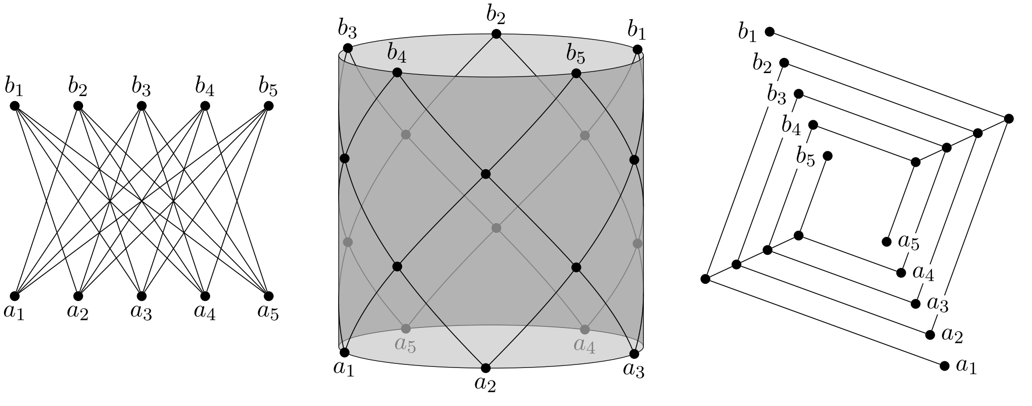

Fig. 1. From left to right: standard example S_5 ($a_i < b_j$ if and only if $i \neq j$); Trotter's poset with planar cover graph containing S_5 as a subposet; Kelly's planar poset containing S_5 as a subposet.

introduced in 1941 by Dushnik and Miller [3] and since then has been one of the most extensively studied parameters in the combinatorics of posets. Much of this research has been focused on understanding when and why dimension is bounded, and this is also the focus of the current paper. The monograph [24] contains a comprehensive introduction to poset dimension theory.

To some extent, dimension for posets behaves like chromatic number for graphs. There is a natural construction of a poset with dimension d, the *standard example* S_d (see Fig. 1), which plays a similar role to the complete graph K_d in the graph setting. Every poset that contains S_d as a subposet must have dimension at least d. On the other hand, there are posets of arbitrarily large dimension not containing S_3 as a subposet, just as there are triangle-free graphs with arbitrarily large chromatic number. Moreover, it is NP-complete to decide whether a poset has dimension at most d for any $d \geq 3$ [28], just as it is for the chromatic number.

These similarities motivated research on how the dimension of a poset depends on its "graphic structure". There are two natural ways of deriving a graph from a poset: the *comparability graph* connects any two comparable elements, while the *cover graph* connects any two elements that are comparable and whose comparability is not implied by other comparabilities and by transitivity of the order. It is customary to include only the cover graph edges in drawings of posets and to describe the "topology" of a poset in terms of its cover graph rather than its comparability graph. This choice has a clear advantage: posets of large height still can have sparse "topology".

The above-mentioned analogy of poset dimension to graph chromatic number suggests that posets with sparse "topology" should have small dimension. In this vein, Trotter and Moore [25] showed that posets whose cover graphs are trees have dimension at most 3. How about planarity—can we expect a property of posets similar to the famous four-color theorem? There is no strict analogy: Trotter [23] and Kelly [15] constructed posets with planar cover graphs that still contain arbitrarily large standard examples as subposets and thus have arbitrarily large dimension (see Fig. 1). Actually, Kelly's

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