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# Generalised Mycielski graphs, signature systems, and bounds on chromatic numbers

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ABSTRACT

We prove that the coindex of the box complex $\mathrm{B}(H)$ of a graph $H$ can be measured by the generalised Mycielski graphs which admit a homomorphism to $H$. As a consequence, we exhibit for every graph $H$ a system of linear equations, solvable in polynomial time, with the following properties: if the system has no solutions, then $\mathrm{coind}(\mathrm{B}(H)) + 2 \leq 3$; if the system has solutions, then $\chi(H) \geq 4$. We generalise the method to other bounds on chromatic numbers using linear algebra.



## 1. Introduction

For any integer $k \geq 2$ and real number $\epsilon \in (0, 1)$, the *Borsuk graph* $B_{k,\epsilon}$ is the graph whose vertices are the points of the $(k-2)$-sphere $S_{k-2} \subseteq \mathbb{R}^{k-1}$, and whose edges join pairs of points $X, Y$ that are "almost antipodal" in the sense that the norm of $X - Y$ is at least $2 - \epsilon$. In [6], Erdős and Hajnal used the Borsuk–Ulam theorem to prove that the chromatic number of $B_{k,\epsilon}$ is $k$. In fact, they proved that the statement $\chi(B_{k,\epsilon}) = k$ is equivalent to the Borsuk–Ulam theorem. For some years this result remained a curiosity

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involving infinite graphs. Then Lovász [8] devised complexes that allow the use of the Borsuk–Ulam Theorem to find lower bounds on chromatic numbers of finite graphs, and used this method to prove the Kneser conjecture on the chromatic number of the Kneser graphs.

Lovász' method inspired many adaptations and developments, giving rise to the field of "topological lower bounds" on the chromatic number of a graph. Our work is inspired by a bound in terms of "coindices of box complexes", specifically

$$\chi(H) \geq \operatorname{coind}(\mathrm{B}(H)) + 2.$$

The relevant definitions of the coindex $\operatorname{coind}(\mathrm{B}(H))$ of the box complex $\mathrm{B}(H)$ of $H$ are well detailed in [9,10,12]. However, our intent is to avoid the topological setting. We will use the following result of Simonyi and Tardos.

**Theorem 1** *([12]). For any graph $H$, $\operatorname{coind}(\mathrm{B}(H)) + 2$ is the largest $k$ such that there exists $\epsilon > 0$ for which $B_{k,\epsilon}$ admits a homomorphism (that is, an edge-preserving map) to $H$.*

This result indeed allows us to restrict our discussion to the field of graphs and homomorphisms: We can alternatively define $\operatorname{coind}(\mathrm{B}(H)) + 2$ *as* the largest $k$ such that there exists an $\epsilon > 0$ for which $B_{k,\epsilon}$ admits a homomorphism to $H$. This viewpoint yields an economy of definitions, but is not necessarily practical for computational purposes. Indeed in many cases a knowledge of simplicial complexes and topological tricks is needed to compute $\operatorname{coind}(\mathrm{B}(H)) + 2$ and effectively bound $\chi(H)$.

Dochtermann and Schultz [4] found finite ("spherical") graphs that play the role of the Borsuk graphs in Theorem 1. In this note we show that the generalised Mycielski graphs can also be used in the same role. We do not know whether this alternative presentation yields effective computations in the general case. However, for low values of $\operatorname{coind}(\mathrm{B}(H))$, our definition indeed leads to practical calculations that can be shown to be conclusive in some cases. The method, in turn, inspires effectively computable lower bounds on the chromatic number of a graph.

## 2. Generalised Mycielski graphs

We will use the following definitions of categorical products, looped paths, cones and generalised Mycielski graphs. The *categorical product* of two graphs $G$ and $G'$ is the graph $G \times G'$ defined by

$$V(G \times G') = V(G) \times V(G'),$$
$$E(G \times G') = \{[(u, u'), (v, v')] : [u, v] \in E(G) \text{ and } [u', v'] \in E(G')\}.$$

We sometimes use directed graphs as factors, and view undirected graphs as symmetric directed graphs. In this case, all square brackets representing edges in the above definition