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Graph parameters from symplectic group invariants

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ABSTRACT

In this paper we introduce, and characterize, a class of graph parameters obtained from tensor invariants of the symplectic group. These parameters are similar to partition functions of vertex models, as introduced by de la Harpe and Jones (1993) [5]. Yet they give a completely different class of graph invariants. We moreover show that certain evaluations of the cycle partition polynomial, as defined by Martin (1977) [15], give examples of graph parameters that can be obtained this way.

1. Introduction

Partition functions of statistical (spin and vertex) models as graph parameters were introduced by de la Harpe and Jones in [5]. Partition functions of spin models include the number of graph homomorphisms into a fixed graph, and they play an important role in the theory of graph limits, cf. [12]. A standard example of the partition function of a vertex model is the number of matchings. Szegedy [25,26] showed that the partition function of any spin model can be realized as the partition function of a vertex model.

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Partition functions of vertex models occur in several different mathematical disciplines. For example as Lie algebra weight systems in the theory of Vassiliev knot invariants cf. [4], as tensor network contractions in quantum information theory [14] and as Holant problems in theoretical computer science [2,3,28].

In [25], Szegedy calls a vertex model an edge-coloring model and we will adopt his terminology here. Let $\mathbb{N}$ include 0, and for $k \in \mathbb{N}$ we denote by $[k]$ the set $\{1, \ldots, k\}$. Let V_k be a k-dimensional vector space over $\mathbb{C}$ and let V_k^* denote its dual space, i.e. the space of linear functions $V_k \to \mathbb{C}$. Following Szegedy [25], for $k \in \mathbb{N}$ we will call $h = (h^n)$, with h^n a symmetric tensor in $(V_k^*)^{\otimes n}$ for each $n \in \mathbb{N}$, a *k-color edge-coloring model* (in [5] it is called a vertex model). We will often omit the reference to k. The *partition function* of h is the graph parameter p_h defined for a graph $G = (V, E)$ by

$$p_h(G) := \sum_{\phi: E \to [k]} \prod_{v \in V} h^{\deg(v)}\Big(\bigotimes_{a \in \delta(v)} e_{\phi(a)}\Big), \tag{1}$$

where for $v \in V$, $\delta(v)$ denotes the set of edges incident with v and where $e_1, \ldots, e_k$ is a basis for V_k. (Note that since $h^{\deg(v)}$ is symmetric the order is irrelevant.)

Starting with the work of Freedman, Lovász and Schrijver [8] and Szegedy [25] a line of research has emerged in which partition function of spin and edge-coloring models have been characterized for several types of combinatorial structures such as, graphs [13,21,6], directed graphs [6,24], virtual link diagrams [18] and chord diagrams [23]. The characterizations of partition functions of edge-coloring models revealed an intimate connection between the invariant theory of the orthogonal and general linear group and these partition functions. However, the symplectic group never showed up. In this paper we will introduce, and characterize, a class of graph parameters related to tensors invariants of the symplectic group that we call *skew-partition functions of edge-coloring models*.

It turns out that these skew-partition functions are most naturally defined for directed graphs, but, surprisingly, we show that when restricted to skew-symmetric tensors, one can in fact define them for undirected Eulerian graphs. These skew-partition functions are related to 'negative dimensional' tensors; see [16] in which Penrose already describes a basic example. For suitable choice of tensors, these skew-partition functions give rise to evaluations of the cycle-partition polynomial (a normalization of the Martin polynomial [15]) at negative even integers. As such, these skew-partition functions play a similar role for the cycle partition polynomial as the number of homomorphisms into the complete graph for the chromatic polynomial.

Besides their connection to the symplectic group, the introduction of skew-partition functions is also motivated by a paper of Schrijver [22]. In [22] Schrijver characterized partition functions of edge-coloring models in terms of rank growth of edge-connection matrices. We need some definitions to state this result. For $k \in \mathbb{N}$, a *k-fragment* is a graph which has k vertices of degree one labeled $1, 2, \ldots, k$. We will refer to an edge incident with a labeled vertex as an *open end*. Let $\mathcal{F}_k$ denote the collection of all k-fragments,