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A stability result for the Katona theorem

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Dedicated to a dear friend, G.O.H. Katona on his 75th birthday

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ABSTRACT

Let n, s be positive integers, $n \geq s + 2$. In 1964 Katona [5] established the maximum possible size of a family of subsets of $\{1, 2, \ldots, n\}$ such that the union of any two members of the family has size of at most s. Katona also proved that the optimal families are unique up to isomorphism. In the present paper we sharpen this result by showing that excluding those optimal families one can get better bounds. These new bounds are best possible.

1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be the standard n-element set and $\mathcal{F} \subset 2^{[n]}$ be a family of subsets. We say that $\mathcal{F}$ has the **s-union property**, or simply that $\mathcal{F}$ is **s-union** if $|F \cup F'| \leq s$ holds for all $F, F' \in \mathcal{F}$.

Definition 1. Let $m(n, s)$ denote the maximum of $|\mathcal{F}|$ over all $\mathcal{F} \subset 2^{[n]}$ having the s-union property.

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Note that $m(n,n) = 2^n$ and that $m(n,n-1) = 2^{n-1}$ was proved already by Erdős–Ko–Rado [1]. They showed also that there are many non-isomorphic $(n-1)$-union families verifying $|\mathcal{F}| = 2^{n-1}$.

The following classical result of Katona determines $m(n,s)$ and all extremal families for $s \le n-2$.

Theorem 1 *(Katona [5]). Let $0 \le s < n$ then (1) or (2) holds:*

(1) $s = 2d$,

$$m(n,s) = \sum_{0 \le i \le d} \binom{n}{i}; \tag{1}$$

(2) $s = 2d+1$,

$$m(n,s) = \sum_{0 \le i \le d} \binom{n}{i} + \binom{n-1}{d}. \tag{2}$$

Moreover for $s \le n-2$, only the Katona families $\mathcal{K}(n,s)$ achieve equality, where

$$\mathcal{K}(n,2d) = \{F \subset [n] : |F| \le d\}$$
$$\mathcal{K}(n,2d+1) = \{F \subset [n] : |F| \le d\} \cup \{F \subset [n] : |F| = d+1,\ y \in F\}$$

where y is an arbitrary but fixed element of $[n]$.

The aim of the present paper is to determine the suboptimal families. That is for fixed n and s we are going to consider s-union families $\mathcal{F}$ satisfying $\mathcal{F} \not\subset \mathcal{K}(n,s)$.

Definition 2. Let us define the families $\mathcal{H}(n,s)$.

$$\mathcal{H}(n,2d) = \{H \subset [n] : |H| \le d\} - \left\{H \in \binom{[n]}{d} : H \cap D = \emptyset\right\} \cup \{D\}$$

where D is a *fixed* $(d+1)$-element subset of $[n]$.

$$\mathcal{H}(n,2d+1) = \{H \subset [n] : |H| \le d+1\} \cup \left\{H \in \binom{[n]}{d+1} : y \in H,\ H \cap D \ne \emptyset\right\} \cup \{D\}$$

where $D \in \binom{[n]}{d+1}$ is fixed and y is a fixed element of $[n] - D$.

Note that $\mathcal{H}(n,s)$ is s-union for both $s = 2d$ and $s = 2d+1$. For $s = 5$, define also

$$\mathcal{T}(n,5) = \{T \subset [n] : |T| \le 2\} \cup \left\{H \in \binom{[n]}{3} : |H \cap [3]| \ge 2\right\}.$$