Journal of Combinatorial Theory, Series B 122 (2017) 877–896

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

The chromatic number of finite type-graphs

Christian Avart [a], Bill Kay [b,1], Christian Reiher [c], Vojtěch Rödl [b,2]

[a] *OtoSense, 3239 El Camino Real, Palo Alto, CA 94306, USA*
[b] *Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA*
[c] *Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany*

ARTICLE INFO

Article history:
Received 9 January 2016
Available online 10 November 2016

Keywords:
Shift graphs
Type-graphs
Chromatic number
Order types of pairs

ABSTRACT

By a *finite type-graph* we mean a graph whose set of vertices is the set of all k-subsets of $[n] = \{1, 2, \ldots, n\}$ for some integers $n \geq k \geq 1$, and in which two such sets are adjacent if and only if they realise a certain order type specified in advance. Examples of such graphs have been investigated in a great variety of contexts in the literature with particular attention being paid to their chromatic number. In recent joint work with Tomasz Łuczak, two of the authors embarked on a systematic study of the chromatic numbers of such type-graphs, formulated a general conjecture determining this number up to a multiplicative factor, and proved various results of this kind. In this article we fully prove this conjecture.

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E-mail addresses: christian.avart@otosense.com (C. Avart), bill.w.kay@gmail.com (B. Kay), Christian.Reiher@uni-hamburg.de (C. Reiher), rodl@mathcs.emory.edu (V. Rödl).

[1] The second author was supported by NCN grant 2012/06/A/STI/00261 and by NSF grant DMS 1301698.
[2] The fourth author was supported by NSF grants DMS 1301698 and 1102086.

http://dx.doi.org/10.1016/j.jctb.2016.10.004
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1. Motivation

Our goal in this article is to analyse the asymptotic behaviour of the chromatic number of certain finite graphs, that are called type-graphs in the sequel. In general the vertex set of such a graph is, for some positive integers $n \geq k$, the collection of all k-element subsets of the set $[n] = \{1, 2, \ldots, n\}$. Whether two such subsets are to be connected by an edge or not is decided solely in terms of the mutual position of their elements or, equivalently, it only depends on the order type that this pair of sets realises. Before defining these type-graphs accurately, we would like to fix notation concerning order types of pairs of ordered sets. In particular, we shall encode such order types as finite sequences consisting of ones, twos, and threes. At first sight, allowing rational numbers in the definition that follows might look unnecessarily general, but it will turn out to be useful at a later occasion.

Definition 1.1. Let X and Y be two finite sets of rational numbers with $|X \cup Y| = \ell$ and $X \cup Y = \{z_1, z_2, \ldots, z_\ell\}$, these elements being listed in increasing order. We say that the *order type* of the pair (X, Y) is the sequence $\tau = (\tau_1, \ldots, \tau_\ell)$ and set $\tau(X, Y) = \tau$ if for every $i \in [\ell]$ we have

$$\tau_i = \begin{cases} 1 & \text{if } z_i \in X \smallsetminus Y\,, \\ 2 & \text{if } z_i \in Y \smallsetminus X\,, \\ 3 & \text{if } z_i \in X \cap Y\,. \end{cases}$$

For example, given $X = \{1, 2, 3, 5\}$ and $Y = \{3, 4, 5\}$ we get $\tau(X, Y) = 11323$. Clearly for any finite sequence τ consisting of ones, twos, and threes there are two finite subsets X and Y of $\mathbb{Q}$ with $\tau = \tau(X, Y)$ and in fact one may even find such sets with $X, Y \subseteq \mathbb{N}$.

The case most relevant for the definition of type-graphs below is $|X| = |Y|$.

Definition 1.2. Consider two nonnegative integers k and ℓ. By a *type* of *width* k and *length* ℓ we mean the order type of a pair (X, Y) with $X, Y \subseteq \mathbb{Q}$, $|X| = |Y| = k$, and $|X \cup Y| = \ell$.

So $\tau = 123312$ is a type of width 4 and length 6 that is realised, e.g., by $X = \{1, 3, 4, 7\}$ and $Y = \{2, 3, 4, 9\}$. It is not hard to observe that in any type of width k and length ℓ there appear $\ell - k$ ones, $\ell - k$ twos, and $2k - \ell$ threes. As a degenerate case we regard the empty sequence $\varnothing$ as an *empty type* of width and length 0. A type is said to be *trivial* if it consists of threes only, or in other words if its width equals its length.

Now we are prepared to define the main objects under consideration in this article.

Definition 1.3. For a nontrivial type τ of width k and an integer $n \geq k$, the *type-graph* $G(n, \tau)$ is the graph with vertex set $\binom{[n]}{k}$ in which two vertices X and Y are declared to be adjacent if and only if we have $\tau(X, Y) = \tau$ or $\tau(Y, X) = \tau$.

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