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# Itineraries for inverse limits of tent maps: A backward view

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ABSTRACT

Previously published admissibility conditions for an element of $\{0,1\}^{\mathbb{Z}}$ to be the itinerary of a point of the inverse limit of a tent map are expressed in terms of forward orbits. We give necessary and sufficient conditions in terms of backward orbits, which is more natural for inverse limits. These backward admissibility conditions are not symmetric versions of the forward ones: in particular, the maximum backward itinerary which can be realised by a tent map mode locks on intervals of kneading sequences.



## 1. Introduction

Inverse limits of tent maps have been much investigated, not only because of their intrinsic interest as topological spaces, but also because they are closely related to other topics in dynamical systems such as hyperbolic attractors and Hénon maps. A recent highlight is the proof by Barge, Bruin, and Štimac of the Ingram Conjecture [2], which states that the inverse limits of distinct tent maps are non-homeomorphic.

Kneading theory is widely used in the study of the dynamics of unimodal maps, and has been extended to and applied in the context of inverse limits of tent maps by several authors (e.g. [3,4]). A key starting point for the application of such symbolic techniques is understanding the *admissibility conditions* under which a sequence of symbols is realised as the itinerary of a point of the inverse limit. In previous works, such admissibility conditions have been adapted from those for kneading theory of unimodal maps, and as

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such are based on the forward itineraries of points. This is somewhat unnatural in the context of inverse limits, where the main focus is on backward orbits.

In this paper we develop admissibility conditions for inverse limits which are based on backward itineraries. One might naïvely expect these conditions to be symmetric versions of the forward ones but, with the exception of certain special cases (tent maps of *irrational* or *rational endpoint* type), this is not the case. The essential content of the forward conditions is that every forward sequence must be less than or equal to the kneading sequence of the tent map $f$, in the unimodal order. For the backward conditions, the kneading sequence is replaced by two sequences, so that backward sequences are bounded by a stepped line. Moreover, these two sequences mode-lock on intervals in parameter space – what changes as the parameter varies within such an interval is the location of the step between the two sequences.

In Section 2 we review the forward admissibility conditions. This theory is well established, but we make some minor modifications which enable us to give admissibility conditions which are strictly necessary and sufficient (Lemmas 3 and 5), which seem not to have appeared before. The basis of the backward admissibility conditions is the stratification of the space of unimodal maps by *height*, a number in $[0, 1/2]$ which is associated to each unimodal map [8]. This theory is reviewed in Section 3.1, before the main results are stated and proved. Theorem 14 gives necessary and sufficient backward admissibility conditions in the symmetric case; Theorem 16 is the analogous result in the non-symmetric case; and Theorem 17 provides a striking illustration of the asymmetry of forward and backward itineraries: the maximum backward itinerary which can be realised by a tent map mode locks on intervals of kneading sequences.

## 2. Forward admissibility

### 2.1. Basic definitions

Throughout the paper, $I = [a, b]$ is a compact interval and $f: I \to I$ is a tent map of slope $\lambda \in (\sqrt{2}, 2)$: that is, there is some $c \in (a, b)$ such that $f$ has constant slope $\lambda$ on $[a, c]$ and constant slope $-\lambda$ on $[c, b]$. Moreover, we assume that $I$ is the dynamical interval (or core) of $f$, so that $f(c) = b$ and $f(b) = a$.

Let $\{0, 1\}^{\mathbb{N}}$ and $\{0, 1\}^{\mathbb{Z}}$ denote the spaces of semi-infinite and bi-infinite sequences over $\{0, 1\}$, with their natural product topologies. We denote elements of the former with lower-case letters, and of the latter with upper-case letters. We write $\sigma: \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ and $\sigma: \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ for the corresponding shift maps. If $S \in \{0, 1\}^{\mathbb{Z}}$, we denote by $\overrightarrow{S}$ and $\overleftarrow{S}$ the elements of $\{0, 1\}^{\mathbb{N}}$ defined by $\overrightarrow{S}_r = S_r$ and $\overleftarrow{S}_r = S_{-1-r}$ for $r \geq 0$: therefore $\overrightarrow{\sigma^r(S)} = S_r S_{r+1} \dots$ and $\overleftarrow{\sigma^r(S)} = S_{r-1} S_{r-2} \dots$ for each $r \in \mathbb{Z}$. We say that $S$ *does not end* $0^\infty$ (respectively *does not start* $0^\infty$) if infinitely many of the entries of $\overrightarrow{S}$ (respectively $\overleftarrow{S}$) are 1.

If $n \geq 1$ then a *word of length* $n$ is an element of $\{0, 1\}^n$. We say that a word $W$ is *even* (respectively *odd*) if it contains an even (respectively odd) number of 1s. If $s \in \{0, 1\}^{\mathbb{N}}$ and $W$ is a word of length $n$, then we write $s = W \dots$ to mean that $s_i = W_i$ for $0 \leq i \leq n - 1$.

We denote by $\preceq$ the *unimodal order* on $\{0, 1\}^{\mathbb{N}}$ (also known as the *parity lexicographical order*), which is defined as follows: if $s$ and $t$ are distinct elements of $\{0, 1\}^{\mathbb{N}}$, then $s \prec t$ if and only if the word $s_0 \dots s_r$ is even, where $r \geq 0$ is least with $s_r \neq t_r$. An element $s$ of $\{0, 1\}^{\mathbb{N}}$ is said to be *shift-maximal* if $\sigma^r(s) \preceq s$ for all $r \geq 0$.

There are several different approaches to assigning itineraries in $\{0, 1\}^{\mathbb{N}}$ to points of $I$ under the action of $f$. Those differences which are not cosmetic are concerned with the straightforward but vexed question of how to code the critical point $c$, and therefore affect the itineraries of only countably many points. One may introduce a third symbol $C$; make an arbitrary choice of 0 or 1 as the code of the critical point; allow either of these symbols, leading to multiple itineraries for certain points – an approach whose ramifications are compounded when the critical point is periodic; or take limits à la Milnor–Thurston [9]. The approach which we adopt here is to code $c$ with a choice of 0 or 1 which depends on $f$ in the case where $c$ is periodic; and to allow either code for $c$ if $c$ is not periodic. This convention, as well as being ideal for our results, has the