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Virtual Special Issue – In honor of Professor Yukihiro Kodama on his 85th birthday

PFA(S)[S] for the masses

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A B S T R A C T

We present S. Todorcevic's method of forcing with a coherent Souslin tree over restricted iteration axioms as a black box usable by those who wish to avoid its complexities but still access its power.

1. Introduction

This paper is dedicated to Professor Y. Kodama, who hosted my very first lecture in Japan more than 35 years ago, which was the start of many fruitful interchanges with Japanese topologists and set theorists.

This note is an expanded version of a talk presented at the recent *First Pan-Pacific International Conference on Topology and Applications*. I thank the organizers for inviting me and for a superb conference. I thank Editor J.E. Vaughan of *Topology and its Applications* for inviting me to write this up for that journal.

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Todorcevic invented his method in 2001 [30] in order to investigate problems not decided by PFA or combinatorial principles such as $\diamondsuit$. A collection of remarkable results has since been obtained using this method by Todorcevic, P. Larson, A. Dow, and the author. The proofs are technically difficult, so just as *Martin's Axiom* is accessible to those who do not understand iterated forcing, it would be nice to be able to apply this method without the difficult forcing. At present, there is no one axiom that will accomplish this, but we can at least list the more important consequences so far of the method, so that they may be further applied.

A *coherent Souslin tree* is a particular kind of very homogeneous Souslin tree. Its exact definition will not concern us here. All we need know is that its existence follows from $\diamondsuit$ [16].

The notations $\mathrm{MA}_{\omega_1}(S)$, $\mathrm{PFA}(S)$, $\mathrm{MM}(S)$ refer to the weaker versions of MA_{ω_1}, PFA, and MM (Martin's Maximum [12]) obtained by restricting only to those posets that preserve the coherent Souslin tree S under countable chain condition, proper, and preserving-stationary-subsets-of-ω_1 forcing, respectively. Notations such as *PFA*$(S)[S]$ *implies* Φ are shorthand for "in any model obtained by forcing with S over a model of $\mathrm{PFA}(S)$, Φ holds".

A heuristic analogy which may be helpful to the reader is to recall that two principal kinds of consequences of MA_{ω_1} are the "combinatorial" ones following from $\mathrm{MA}_{\omega_1}(\sigma$-centred), and the "Souslin-type" ones [15]. Souslin-type ones are those that imply Souslin's Hypothesis, such as *there are no compact S-spaces*, *there are no first countable L-spaces*, *all Aronszajn trees are special*, etc. In [15] we showed that the failure of Souslin's Hypothesis was consistent with $\mathrm{MA}_{\omega_1}(\sigma$-centred); in models of form e.g. $\mathrm{PFA}(S)[S]$ one obtains most of the Souslin-type consequences of PFA, but many of the combinatorial ones fail, indeed $\mathfrak{p} = \aleph_1$.

From the point of view of a topologist, the name of the game is to use consequences of MA_{ω_1}, PFA, or MM proven – usually with more difficult proofs – from $\mathrm{MA}_{\omega_1}(S)[S]$, $\mathrm{PFA}(S)[S]$, or $\mathrm{MM}(S)[S]$, and combine them with consequences failing under MA_{ω_1}, PFA, MM, but holding in models of $\mathrm{PFA}(S)[S]$, etc. More particularly, consequences such as normality implying collectionwise Hausdorffness for certain spaces – consequences of $V = L$ – have been shown to hold in some of these models. It is not yet clear what other kinds of useful consequences of $V = L$ might hold in these models. The collectionwise Hausdorff ones have been particularly fruitful; they are actually of more than topological applicability, since they can be translated into uniformization of ladder systems or freeness of Whitehead groups – see Larson–Tall [17].

2. Some consequences

To hold the reader's attention, let us mention some important results obtained by this method. No other method is known to prove the consistency of the conclusions.

1. *$MA_{\omega_1}(S)[S]$ implies that if X is a compact space with X^2 hereditarily normal, then X is metrizable.* Katětov had proved this in ZFC for X^3 hereditarily normal and naturally asked about X^2. 50 years later, P. Larson and Todorcevic solved the problem [19]. Consistent counterexamples had earlier been constructed by G. Gruenhage and P. Nyikos [14].
2. *There is a model of $MA_{\omega_1}(S)[S]$ in which locally compact, perfectly normal spaces are paracompact.* There are many consistent counterexamples, e.g. MA_{ω_1} implies the Cantor tree over a Q-set is a counterexample; $\diamondsuit$ implies Ostaszewski's space is a counterexample, etc. The problem of whether it was consistent there are no counterexamples was raised by S. Watson [32], [33]. Larson and Tall [17] constructed the required model of $\mathrm{PFA}(S)[S]$; A. Dow and Tall managed to drop the large cardinal [6].
3. *There is a model of* $\mathrm{MA}_{\omega_1}(S)[S]$ *in which every hereditarily normal manifold of dimension* > 1 *is metrizable.* The problem of the existence of such a model was raised by Nyikos in 1983 [23] and solved by Dow and Tall 30 years later in [5]. Although they don't bother to get $\mathrm{MA}_{\omega_1}(S)[S]$, their model could be tweaked to be of that form, with additional consequences of PFA thrown in.